AMS-311. Spring 2005. Homework 9.
Topics: Limit Theorems.

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1). Imagine that on a particular roulette wheel, $P(\text{WIN}) = 18/37$. If you play 100 games, find the probability that you win at least half of them. Hint: solve using CLT.

2). Let $X_1, \dots, X_{10}$ be independent random variables, uniformly distributed over the unit interval $[0, 1]$.

(a) Estimate $P(X_1 + \dots + X_{10} \geq 7)$ using the Markov inequality.

(b) Repeat part (a) using the Chebyshev inequality.

(c) Repeat part (a) using the central limit theorem.

3). The test scores of 900 students had the following sample statistics:

Mean: 83; Variance: 36

Use Chebyshevs inequality to bound the probability that a randomly selected student received a test score between 71 and 95 inclusive. Is it likely that at least 600 students scored between 71 and 95 inclusive? Why or why not?

4). The weight of a detail W, is a continuous random variable described by the probability density function

$$f_W(w) = \begin{cases} 0, & w \leq 1 \\ w-1, & 1 \leq w \leq 2 \\ 3-w, & 2 \leq w \leq 3 \\ 0, & 3 \leq w. \end{cases}$$

(a) What is the probability that 102 details weigh more than 200?

(b) What is the smallest integer n for which the total weight of n details will exceed 200 with probability 0.990?

5). We are laying out 25 plastic planks lengthwise, trying to make a path of about 1000 meters. The plastic planks are made in molds, and any variation in the lengths of the planks is due entirely to variation between different molds. The length in meters, X, of any particular mold used for making planks is independent of the length of all other molds. X is uniformly distributed between $40 - \sqrt{3}$ and $40 + \sqrt{3}$ meters. X has an expected value of 40 meters and a standard deviation of 1 meter. What is the probability that the resulting path will be within 1000 ± 7.5 meters if we use 25 planks ...

(a) ... all made from the same mold?

(b) ... each made from a different mold?

6). Random variable X takes on experimental values of -8, 0, and 8 with probabilities of 1/8, 6/8, and 1/8, respectively. T_n is the sum and A_n is the average of n independent experimental values of X (i.e., T_{100} is the sum of 100 independent experimental values of X.)

(a) Evaluate the expectations and variances for T_n and A_n.

(b) Provide a numerical approximation for the quantity:

$$Q = P(|T_{100} - \mathbf{E}[T_{100}]| \geq 32)$$

7). Define X as the height in meters of a randomly selected Canadian, where the selection probability is equal for each Canadian, and denote $\mathbf{E}[X]$ by h. George is interested in estimating h. Because he is sure that no Canadian is taller than 3 meters, George decides to use 1.5 meters as a conservative (large) value for the standard deviation of X. To estimate h, George averages the heights of n Canadians that he selects at random; he denotes this quantity by H.

(a) In terms of h and Georges 1.5 meter bound for the standard deviation of X, determine the expectation and standard deviation for H.

(b) Help George by calculating a minimum value of n (with $n > 0$) such that the standard deviation of Georges estimator, H, will be less than 0.01 meters.

(c) Say George would like to be 99% sure that his estimate is within 5 centimeters of the true average height of Canadians. Using the Chebyshev inequality, calculate the minimum value of n that will make George happy.

(d) If we agree that no Canadians are taller than three meters, why is it correct to use 1.5 meters as an upper bound on the standard deviation for X, the height of any Canadian selected at random?