AMS-311. Spring 2005. Homework 8.
Topics: Markov Chains.

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1). A discrete-time Markov chain with seven states has the following transition probabilities:

$$p_{ij} = \begin{cases} 0.5, & (i,j) = (3,2), (3,4), (5,6) \text{ and } (5,7) \\ 1, & (i,j) = (1,3), (2,1), (4,5), (6,7) \text{ and } (7,5) \\ 0, & \text{otherwise} \end{cases}$$

In the questions below, we let X_k be the state of the Markov process at time k.

(a) Give a pictorial representation of the discrete-time Markov chain.

(b) For what values of n is the probability

$$r_{15}(n) = P(X_n = 5 | X_0 = 1) > 0?$$

(c) What is the set of states $A(i)$ that is accessible from state i, for each $i = 1, 2, \ldots, 7$?

(d) Identify which states are transient and which states are recurrent. For each recurrent class, state whether it is periodic (and give the period) or aperiodic.

(e) What is the minimum number of transitions with nonzero probability that must be added so that all seven states form a single recurrent class?

2). Out of the d doors of my house, suppose that in the beginning $k > 0$ are unlocked and $d - k$ are locked. Every day, I use exactly one door, and I am equally likely to pick any of the d doors. At the end of the day, I leave the door I used that day locked.

(a) Show that the number of unlocked doors at the end of day n, L_n, evolves as the state in a Markov process for $n \geq 1$. Write down the transition probabilities p_{ij} .

(b) List transient and recurrent states.

(c) Is there an absorbing state? How does $r_{ij}(n)$ behave as $n \to \infty$?

(d) Now, suppose that each day, if the door I pick in the morning is locked, I will leave it unlocked at the end of the day, and if it is initially unlocked, I will leave it locked. Repeat parts (a)-(c) for this strategy.

3). The Stony Brook football team's performance in any given game is very much correlated to its morale. In fact, if the team has won the past two games, then it has a .7 probability of winning the next game. If it lost the last game but won before that, it has a .4 probability of winning. If it won its last game but lost before that it has a .5 probability of winning, and finally if it lost the last two games it has only a .2 probability of winning the next game. Assume that the above details the complete correlation between the history of victories and defeats, and the future performance.

(a) Define with a Markov chain that models the above process. Remember that the chain must have the Markov property.

(b) Find the long run probability that the Stony Brook football team will win its next game.

4). At the Harriman Cafe, there is only one cashier. Due to the limited space, she allows only M customers to line before her at any time. If a customer finds there are M customers there including the one being served by the cashier, he will leave the Cafe immediately. Every minute, exactly one of the following occurs:

- one new customer arrives with probability p;
- one existing customer leaves with probability kq, where k is the number of customers in the House; or
- no new customer arrives and no existing customer leaves with probability $1 - p - kq$ if there is at least one customer in the Cafe, and with probability 1 - p otherwise.

(a) This problem can be modeled as a birth-death process. Define appropriate states and draw the transition probability graph.

(b) After the Cafe has been open for a long time, you walk into the Cafe. Calculate how many customers you expect to see in line.

5). For a series of dependent trials, the probability of success on any given trial is given by $(k+1)/(k+3)$, where k is the number of successes in the previous three trials. Define a state description and a set of transition probabilities which allow this process to be described as a Markov chain. Draw the state transition diagram. Try to use the smallest possible number of states.

6). You are reading a challenging technical book that has three chapters whose (abbreviated) titles are F, G and H; the book will take you a lifetime to master, starting today (day 0). If you're reading chapter F on a particular day, there's a probability β that you are still reading chapter F the next day, and otherwise you will be reading chapter G. If you're reading chapter G one day, you will be reading chapter H the next day with probability p, otherwise you'll be reading chapter F. And if you're reading chapter H one day, there's a probability p that you will still be with chapter H the next day, otherwise you will go to chapter G.

(a) Draw a representation of your reading process as a Markov chain.

(b) If $\beta = 1$, what are the (approximate) probabilities that you will be reading chapters F, G and H respectively at some arbitrarily chosen day many months from now?

(c) Repeat the previous part if $\beta < 1$.

(d) Assume $\beta < 1$. If you're in chapter G on a particular day, what is the expected number of additional days up to and including the day you are first back at chapter F? Similarly, if you are in chapter H on a particular day, what is the expected number of additional days up to and including the day you are first back at chapter F?

(e) Again, assume $\beta < 1$. What is the expected number of days until you first repeat chapter F, given that you start in chapter F on day 0?