AMS-311. Spring 2005. Homework 7.
Topics: Bernoulli processes. Poisson processes.

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1). You are visiting the rainforest, but unfortunately your insect repellent has run out.

(a) As a result, at each second, a mosquito lands on your neck with probability 0.2.

i. Whats the PMF for the time until the first mosquito lands on you?

ii. Whats the expected time until the first mosquito lands on you?

iii. What if you weren't bitten for the first ten seconds – what would be the expected time until the first mosquito lands on you (from time $t = 10$)?

(b) Instead, imagine the rainforest had only one mosquito, which arrived in the following way: the time of arrival is exponentially distributed with $\lambda = 0.2$.

i. Whats the expected time until the first mosquito lands on you?

ii. What if you weren't bitten for the first ten seconds - what would be the expected time until the first mosquito lands on you (from time $t = 10$)?

2). Based on your understanding of the Poisson process, determine the numerical values of a and b in the following expression and explain your reasoning:

$$\int_t^\infty \frac{\lambda^6 \tau^5 e^{-\lambda\tau}}{5!}\, d\tau = \sum_{k=a}^{b} \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

3). Transmitters A and B independently send messages to a single receiver in a Poisson manner with average message arrival rates of λ_A and λ_B, respectively. All messages are so brief that we may safely assume that they occupy only single points in time. The number of words in every message, regardless of its transmitting source, may be considered to be an independent experimental value of random variable W with PMF

$$p_W(w) = \begin{cases} 2/6, & w = 1 \\ 3/6, & w = 2 \\ 1/6, & w = 3 \\ 0, & \text{otherwise} \end{cases}$$

(a) What is the probability that, during an interval of duration t, a total of exactly nine messages will be received?

(b) Let N be the total number of words received during an interval of duration t. Determine the expected value for random variable N.

(c) Determine the PDF for X, the time from $t = 0$ until the receiver has received exactly eight three-word messages from transmitter A.

(d) What is the probability that exactly eight of the next twelve messages received will be from transmitter A?

4). (a) Shuttles depart from New York to Boston every hour on the hour. Passengers arrive according to a Poisson process of rate λ per hour. Find the expected number of passengers on a shuttle. (Ignore issues of limited seating.)

(b) Now suppose that the shuttles are no longer operating on a deterministic schedule, but rather their interdeparture times are independent and exponentially distributed with rate μ per hour. Find the PMF for the number of shuttles arriving in one hour.

(c) Let us define an "event" in the airport to be either the arrival of a passenger, or the departure of a plane. With the same assumptions as in (b) above, find the expected number of "events" that occur in one hour.

(d) If a passenger arrives at the gate, and sees 2λ people waiting, find his/her expected time to wait until the next shuttle.

(e) Find the PMF for the number of people on a shuttle.

5). All ships travel at the same speed through a wide canal. Eastbound ship arrivals at the canal are a Poisson process with an average arrival rate λ_E ships per day. Westbound ships arrive as an independent Poisson process with average arrival rate λ_W per day. An indicator at a point in the canal is always pointing in the direction of travel of the most recent ship to pass it. Each ship takes t days to traverse the length of the canal.

(a) Given that the pointer is pointing west:

i. What is the probability that the next ship to pass it will be westbound?

ii. What is the PDF for the remaining time until the pointer changes direction?

(b) What is the probability that an eastbound ship will see no westbound ships during its eastward journey through the canal?

(c) We begin observing at an arbitrary time. Let V be the time we have to continue observing until we see the seventh eastbound ship. Determine the PDF for V.

6). [**Extra credit**] Suppose that W, the amount of moisture in the air on a given day, is a gamma random variable with parameters (t, β). That is, its density is

$$f(w) = \frac{\beta e^{-\beta w}(\beta w)^{t-1}}{\Gamma(t)}, \quad w > 0.$$

Suppose also that given that $W = w$, the number of accidents during that day, call it N, has a Poisson distribution with mean w. Show that the conditional distribution of W given $N = n$ is the gamma distribution with parameters $(t + n, \beta + 1)$.

Note that $\Gamma(z)$ is the gamma function defined as:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt.$$

Can you simplify $\Gamma(z)$ when z is a positive integer? (don't say "no")