AMS-311. Spring 2005. Homework 5.
Topics: Continuous random variables, PDFs, Expectation, Variance.

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1). Let X and Y be the random variables with the following joint PDF:

$$f_{X,Y}(x,y) = \begin{cases} c, & x \geq 0, y \geq 0, \text{ and } x+y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Let A be the event $\{X \geq 0.5\}$ and let B be the event $\{X > Y\}$. Draw a sketch and answer the following questions:

(a) Are X and Y independent? Explain.

(b) Calculate the numerical value of c.

(c) Calculate the probability $P(B|A)$.

(d) Make a clearly labeled sketch of the conditional PDF $f_{X|Y}(x|0.5)$.

(e) Given that $Y = 0.5$, evaluate the conditional expectation and the conditional variance of X.

(f) Make a clearly labeled sketch of the conditional PDF $f_{X|B}(x|B)$.

2). One of two wheels of fortune, A and B, is selected by the flip of a fair coin, and the wheel chosen is spun once to determine the experimental value of random variable X. Random variable Y, the reading obtained with wheel A, and random variable W, the reading obtained with wheel B, are described by the PDFs

$$f_Y(y) = \begin{cases} 1, & 0 < y \leq 1 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad f_W(w) = \begin{cases} 3, & 0 < w \leq 1/3 \\ 0, & \text{otherwise} \end{cases}$$

If we are told the experimental value of X was less than $1/4$, what is the conditional probability that wheel A was the one selected?

3). Random variables X and Y are independent and are described by the probability density functions $f_X(x)$ and $f_Y(y)$:

$$f_X(x) = \begin{cases} 1, & 0 < x \leq 1 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad f_Y(y) = \begin{cases} 1, & 0 < y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Stations A and B are connected by two parallel message channels. One message from A to B is sent over each of the channels at the same time. Random variables X and Y represent the message delays in hours over parallel channels 1 and 2, respectively.

A message is considered "received" as soon as it arrives on any one channel and it is considered "verified" as soon as it has arrived over both channels.

(a) Determine the probability that a message is received within 15 minutes after it is sent.

(b) Determine the probability that the message is received but not verified within 15 minutes after it is sent.

(c) Let T represent the time in hours between transmission at A and verification at B. Determine the CDF $F_T(t)$, and then differentiate it to obtain the PDF $f_T(t)$.

(d) If the attendant at B leaves for a 15-minute coffee break right after the message is received, what is the probability that he is present at the proper time for verification?

(e) The management wishes to have the maximum probability of having the attendant present for both reception and verification. Would they do better to let him take his coffee break as described above or simply allow him to go home 45 minutes after transmission?

4). A signal $s = 2$ is transmitted from a satellite but is corrupted by noise, so that the received signal is $X = s + W$. When the weather is good, which happens with probability 2/3, W is normal with zero mean and variance 1. When the weather is bad, W is normal with zero mean and variance 3^2. In the absence of any weather information, find the PDF of X and calculate the probability that X is between 1 and 3.

5). Let X be uniformly distributed on $[0, 1]$, and let Y be exponentially distributed with parameter λ.

(a) Find $P(X \geq \mathbf{E}[X])$.

(b) Given $X \geq t$, determine the conditional PDF of $X - t$.

(c) Find $P(Y \geq \mathbf{E}[Y])$.

(d) Given $Y \geq t$, determine the conditional PDF of $Y - t$.

6). Let X_1, X_2 and X_3 be three independent, continuous random variables with the same distribution. Given that X_2 is smaller than X_3, what is the conditional probability that X_1 is smaller than X_2?

7). **[Extra Credit]** Three points A, B, C are chosen independently at random on the circumference of a circle. Let $b(x)$ be the probability that at least one of the angles of the triangle ABC exceeds $x\pi$. Show that

$$b(x) = \begin{cases} 1 - (3x-1)^2, & \frac{1}{3} \leq x \leq \frac{1}{2} \\ 3(1-x)^2, & \frac{1}{2} \leq x \leq 1 \end{cases}$$