AMS-311. Spring 2005. Homework 4.
Topics: Expectation, Variance. Joint PMFs. Conditioning, Independence.

Andrei Antonenko

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1). Random variables X and Y have the joint PMF

$$p_{X,Y}(x,y) = \begin{cases} cxy, & x \in \{1,2,4\} \text{ and } y \in \{1,3\} \\ 0, & \text{otherwise} \end{cases}$$

(a) What is the value of the constant c?

(b) What is $P(Y < X)$?

(c) What is $P(Y > X)$?

(d) What is $P(Y = X)$?

(e) What is $P(Y = 3)$?

(f) Find the marginal PMFs $p_X(x)$ and $p_Y(y)$.

(g) Find the expectations $\mathbf{E}[X]$ and $\mathbf{E}[Y]$.

(h) Find the variances $\mathbf{var}(X)$ and $\mathbf{var}(Y)$.

2). At his workplace, the first thing George does every morning is to go to the supply room and pick up either one, two, or three pens, with each of these possibilities being equally likely. If he receives three pens, he does not return to the supply room again that day. If he receives only one or two pens, he will make one additional trip to the supply room, where he again picks up one, two, or three pens, with equal probability. Note: The number of pens taken in one trip will not affect the number of pens taken in any other trip. Evaluate:

(a) $P(A)$, where A is the event that George picks up a total of 3 pens on a given day.

(b) $P(B|A)$, where B is the event that he visited the supply room twice on the day in the question.

(c) $\mathbf{E}[N]$, and $\mathbf{E}[N|C]$, where N is a total number of pens George picks up any given day, and C is the event that $\{N > 3\}$.

(d) $\mathbf{var}(N|C)$ – conditional variance of N given C.

(e) $P(D)$, where D is the event that he receives more than three pens on each of the next 16 days.

A TREE DIAGRAM can be of a great help in this problem!

3). Professor May B. Right often makes mistakes in her science class. She answers each of her students questions incorrectly with probability 1/4, independently of other questions. In each lecture May is asked 1, 2, or 3 questions with equal probability 1/3.

(a) What is the probability that May gives wrong answers to all the questions she is asked in a given lecture?

(b) Given that May gave wrong answers to all the questions she was asked in a given lecture, what is the probability that she was asked three questions?

(c) Let X and Y be the number of questions May is asked and the number of questions she answers correctly in a lecture, respectively. What are the mean and variance of X and of Y ?

(d) Find the joint PMF of X and Y. Give the answer as a table.

(e) To encourage questions in Mays class, Mays college adopts an unusual incentive policy and offers a bonus of $10X + 20Y$ dollars to May. What is the expected value of the bonus.

(f) Mays semester has 20 lectures. Let Z be the total number of questions she answers wrong in a semester. What is the mean of Z?

4). **[Extra Credit]** Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables. Find the values of a and b that will make the following formula true:

$$\mathbf{E}\left[(X_1 + \cdots + X_n)^2\right] = a\mathbf{E}\left[X_1^2\right] + b(\mathbf{E}\left[X_1\right])^2.$$

5). **[Extra Credit]** Consider a square grid, S, of dimension $n \times n$.

(a) How many quadrilaterals (rectangles and squares) are in S?

(b) If I pick a quadrilateral at random, what is the probability it has dimension $k \times j$ (or $j \times k$), where $k, j \leq n$?