AMS-311. Spring 2005. Homework 3. Topics: Counting, Discrete random variables, PMFs, Expectations.

Andrei Antonenko

Due Feb 18, 2005

1). The hats of n persons are thrown into a box. The persons then pick up their hats "at random" (i.e., so that every assignment of the hats to the persons is equally likely). What is the probability that exactly $n-2$ persons pick their own hat?

2). Alice plays with Bob the following game. First Alice randomly chooses 4 cards out of a 52-card deck, memorizes them, and places them back into the deck. Then Bob randomly chooses 8 cards out of the same deck. Alice wins if Bobs cards include all cards selected by her. What is the probability of this happening?

3). I have just completed n independent tosses of a coin which comes heads with probability p.

(a) Given that I got heads exactly once, what is the probability that my first toss came heads?

(b) Given that I got heads exactly twice, what is the probability that my first toss came heads?

(c) Given that I got exactly 7 heads, what is the probability that exactly three out of the first four tosses were heads?

4). Twenty five black pebbles are arranged in five rows of five pebbles each. We choose five of these pebbles at random and color them red.

(a) What is the probability that all the red pebbles lie in different rows?

(b) What is the probability that all the red pebbles lie in different rows and in different columns?

5). (a) Let X be a random variable that takes nonnegative integer values. Show that

$$\mathbf{E}[X] = \sum_{k=1}^{\infty} P(X \geq k).$$

(Hint: Start with the above formula and interchange the order of the summation, after representing $P(X \geq k) = \sum_{i=k}^{\infty} P(X = k)$)

(b) Use the formula in (5a) to compute the expectation of a random variable Y with the geometric PMF

$$p_Y(k) = p(1-p)^{k-1}, k = 1, 2, \ldots.$$

(Here p is a constant between 0 and 1.)

6). Two fair three-sided dice are rolled simultaneously. Let X be the sum of the two rolls.

(a) Calculate the PMF, the expected value, and the variance of X.

(b) Calculate the PMF of X^2.

7). **[Extra Credit]** A candy factory has an endless supply of red, orange, yellow, green, blue, and violet jelly beans. The factory packages the jelly beans into jars of 100 jelly beans each. One possible color distribution, for example, is a jar of 56 red, 22 yellow, and 22 green jelly beans. As a marketing gimmick, the factory guarantees that no two jars have the same color distribution. What is the maximum number of jars the factory can produce?