AMS-311. Spring 2005. Homework 1. Topics: Set Theory, Probability Axioms, Basics of conditional probability

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1). Of the students in a class, 60% love coffee, 70% live in dorms, and 25% fall into neither group. Determine the probability that a randomly selected student loves coffee or lives in a dorm, but not both.

2). Express each of the following events in terms of the events A, B, and C, as well as the operations of complementation, union, and intersection:

(a) at least one of the events A, B, C occurs;

(b) at most one of the events A, B, C occurs;

(c) none of the events A, B, C occurs;

(d) all three events A, B, C occur;

(e) exactly one of the events A, B, C occurs;

(f) events A and B occur, but not C.

Draw corresponding Venn's diagrams.

3). For three tosses of a fair coin determine the probability of

(a) the sequence HHH;

(b) a total result of two heads and one tail;

(c) an outcome "more tails than heads".

4). Mary and John each choose at random a number between zero and two. We assume a uniform probability law (the probability of the event is proportional to its area). Find the probabilities of the following events:

(a) Mary's number is greater than 1/2.

(b) At least one number is greater than 1/2.

(c) Two numbers are equal.

5). Prove the following **Bonferroni's inequality:**

$$P(A \cap B) \geq P(A) + P(B) - 1.$$

Notice, that the inequality can be generalized to the case of n events (just remember it, you don't need to prove it!)

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) \geq P(A_1) + P(A_2) + \cdots + P(A_n) - (n-1)$$

6). We roll two fair 6-sided dice. Each of the 36 possible outcomes is assumed to be equally likely.

(a) Find the probability that doubles are rolled.

(b) Given that roll results in a sum of 4 or less, find the conditional probability that doubles are rolled.

(c) Given that the two dice land on different numbers, find the conditional probability that at least one die roll is a 6.

7). A test for a certain rare disease is assumed to be correct 95% of the time: if a person has a disease, the test results are positive with probability 0.95, and if the person doesn't have the disease, the test results are negative with probability 0.95. A random person drawn from the population has probability 0.001 of having the disease. Given that the person just tested positive, what is the probability of having the disease?

8). Let $A \subset \Omega$ and $B \subset \Omega$ be the events from some sample space Ω. Show that $P(A \cap B|B) = P(A|B)$, assuming that $P(B) > 0$.

9). [**Extra credit**] **Monty Hall problem.** The Prince is told that the beautiful Princess is equally likely to be found behind any one of three closed doors in front of him (there is a mad tigers behind two others!) The Prince points to one of the doors. A servant opens for him one of the remaining two doors, after making sure that the Princess is not behind it. At this point the Prince can stick to his initial choice, or switch to the other unopened door. The Prince marries the Princess if it is behind his final choice of a door. Consider two following strategies for the Prince:

(A) Stick to his initial choice.

(B) Switch to the other unopened door.

What is his best strategy?