Lecture 32

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1 Expected time to Absorption

In the previous section we figured out that for any Markov chain with several absorbing states, the probability of finally getting absorbed in one of them is equal to 1, and we figured out the way to compute the probabilities of being absorbed by some particular state given the initial state. Here we will figure out how to compute the expected number of steps before absorption takes place. In particular, we will compute

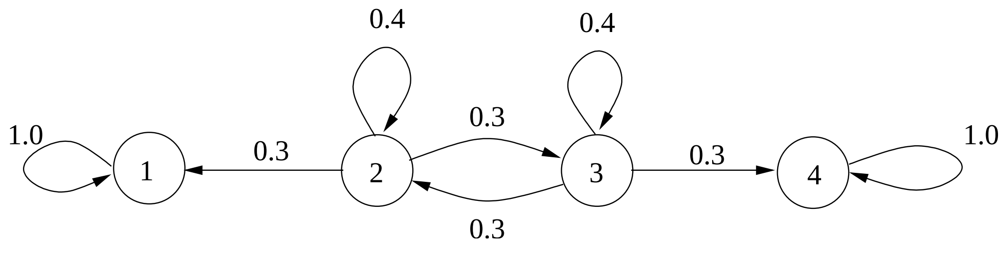
$$\mu_i = \mathbf{E}\left[\text{number of transitions until absorption starting from } i\right]. \tag{1}$$

We can derive equations for the μ_i by using the total expectation theorem. We argue that the time to absorption starting from a transient state i is equal to 1 plus the expected time to absorption starting from the next state, which is j with probability p_{ij} . This leads to a system of linear equations which is stated below. It turns out that these equations have a unique solution, but the argument for establishing this fact is beyond our scope.

$$\mu_i = 0 \qquad \text{if } i \text{ is absorbing state} \tag{2}$$

$$\mu_i = 1 + \sum_{j=1}^{m} p_{ij}\mu_j \qquad \text{if } i \text{ is transient} \tag{3}$$

Example 1.1. Consider fly and spiders example from one of the previous lectures. The graph for the corresponding Markov chain is given on the following picture:



The system of the equations to determine μ_i's is the following:

$$\begin{aligned}
\mu_1 &= 0\\
\mu_4 &= 0\\
\mu_2 &= 1 + 0.3\mu_1 + 0.4\mu_2 + 0.3\mu_3\\
\mu_3 &= 1 + 0.3\mu_2 + 0.4\mu_3 + 0.3\mu_4
\end{aligned}$$

since 1 and 4 are absorbing states, and 2 and 3 are transient. Substituting zero values of μ_1 and μ_4 into last two equations, we get the following system:

$$\mu_2 = 1 + 0.4\mu_2 + 0.3\mu_3$$
$$\mu_3 = 1 + 0.3\mu_2 + 0.4\mu_3$$

solving which we get:

$$\mu_2 = \mu_3 = \frac{10}{3}.$$

2 Mean first passage times

The same idea used to calculate the expected time to absorption can be used to calculate the expected time to reach a particular recurrent state, starting from any other state. Throughout this subsection, we consider a Markov chain with a single recurrent class. We focus on a special recurrent state s, and we denote by t_i the mean first passage time from state i to state s, defined by

$$t_i = \mathbf{E}\left[\text{number of transitions to reach } s \text{ for the first time, starting from } i\right]. \tag{4}$$

The transitions out of state s are irrelevant to the calculation of the mean first passage times. We may thus consider a new Markov chain which is identical to the original, except that the special state s is converted into an absorbing state (by setting $p_{ss} = 1$, and $p_{sj} = 0$ for all $j \neq s$). We then compute t_i as the expected number of steps to absorption starting from i, using the formulas given earlier in this section. We have

$$t_i = 1 + \sum_{j=1}^{m} p_{ij} t_j, \qquad \text{for all } i \neq s, \tag{5}$$

$$t_s = 0 \qquad . \tag{6}$$

This system of linear equations can be solved for the unknowns t_i, and is known to have a unique solution.

The above equations give the expected time to reach the special state s starting from any other state. We may also want to calculate the mean recurrence time of the special state s, which is defined as

$$t_s^* = \mathbf{E}\left[\text{number of transitions up to the first return to } s \text{ starting from } s\right] \tag{7}$$

We can obtain t_s^*, once we have the first passage times t_i, by using the equation

$$t_s^* = 1 + \sum_{j=1}^{m} p_{sj} t_j. \tag{8}$$

To justify this equation, we argue that the time to return to s, starting from s, is equal to 1 plus the expected time to reach s from the next state, which is j with probability p_{sj}. We then apply the total expectation theorem.

Example 2.1. Consider the Markov chain with two states 1 and 2 and the following transition probabilities:

$$p_{11} = 0.8, \qquad p_{12} = 0.2,$$
$$p_{21} = 0.6, \qquad p_{22} = 0.4.$$

Let us focus on state $s = 1$ and calculate the mean first passage time to state 1, starting from state 2. We have

$$t_1 = 0 \quad \text{and} \quad t_2 = 1 + p_{21}t_1 + p_{22}t_2 = 1 + 0.4t_2,$$

from which

$$t_2 = \frac{1}{0.6} = \frac{5}{3}.$$

The mean recurrence time to state 1 is given by

$$t_1^* = 1 + p_{11}t_1 + p_{12}t_2 = 1 + 0 + 0.2 \cdot \frac{5}{3} = \frac{4}{3}.$$