Lecture 31

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1 Absorption probabilities

In this section we will study when the Markov chain enters its absorbing state, and with which probabilities. Assume there is a Markov chain and we start from some transient state. Eventually, we will end up in either one of the recurrent classes, or in some absorbing state, and never leave it. If we enter the recurrent class ar an absorbing state, we are bound to stay there forever, and therefore the subsequent behavior of the Markov chain is not important.

If there is a unique absorbing state k, its steady-state probability is 1 (because all other states are transient and have zero steady-state probability), and will be reached with probability 1, starting from any initial state. If there are multiple absorbing states, the probability that one of them will be eventually reached is still 1, but the identity of the absorbing state to be entered is random and the associated probabilities may depend on the starting state. In the sequel, we fix a particular absorbing state, denoted by s, and consider the absorption probability a_i that s is eventually reached, starting from i:

$$a_i = P(X_n \text{ eventually becomes equal to the absorbing state } s | X_0 = i). \tag{1}$$

Absorption probabilities can be obtained by solving a system of linear equations, as indicated below.

Consider a Markov chain in which each state is either transient or absorbing. We fix a particular absorbing state s. Then, the probabilities a_i of eventually reaching state s, starting from i, are the unique solution of the equations

$$a_s = 1, \tag{2}$$

$$a_i = 0, \text{ for all absorbing } i \neq s, \tag{3}$$

$$a_i = \sum_{j=1}^{m} p_{ij} a_j, \text{ for all transient states } i \tag{4}$$

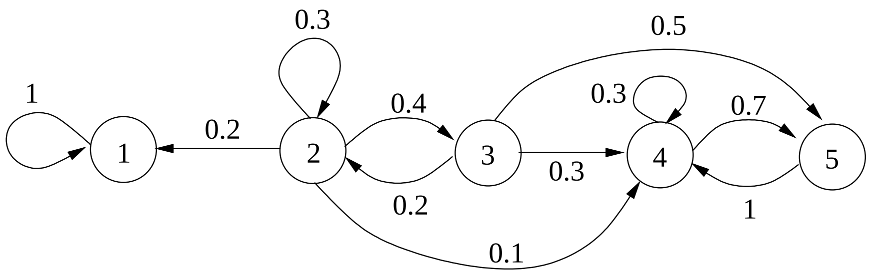
In matrix form the last set of equations (4) can be written as follows:

$$\begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \cdots \\ a_m \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \cdots \\ a_m \end{pmatrix} \tag{5}$$

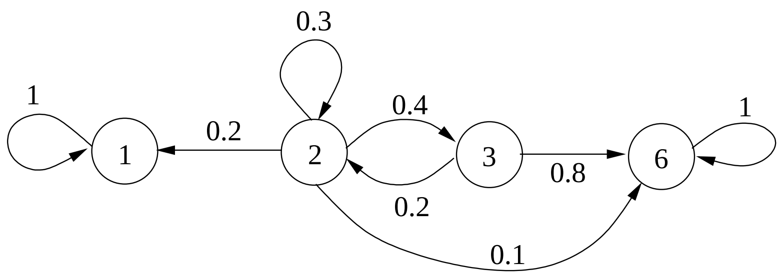
The equations $a_s = 1$, and $a_i = 0$, for all absorbing $i \neq s$, are evident from the definitions. To verify the remaining equations, we argue as follows. Let us consider a transient state i and let A be the event that state s is eventually reached. We have

$$\begin{aligned} a_i &= P(A|X_0 = i) \\ &= \sum_{j=1}^{m} P(A|X_0 = i, X_1 = j)P(X_1 = j|X_0 = i) \\ &= \sum_{j=1}^{m} P(A|X_1 = j)p_{ij} \\ &= \sum_{j=1}^{m} a_j p_{ij} \end{aligned}$$

Example 1.1. Consider the Markov chain shown in the following figure.



We would like to calculate the probability that the state eventually enters the recurrent class $\{4, 5\}$ starting from one of the transient states. For the purposes of this problem, the possible transitions within the recurrent class $\{4, 5\}$ are immaterial. We can therefore lump the states in this recurrent class and treat them as a single absorbing state (call it state 6); see next figure.



It then suffices to compute the probability of eventually entering state 6 in this new chain.

The absorption probabilities a_i of eventually reaching state $s = 6$ starting from state i, satisfy the following equations:

$$\begin{aligned} a_2 &= 0.2a_1 + 0.3a_2 + 0.4a_3 + 0.1a_6, \\ a_3 &= 0.2a_2 + 0.8a_6. \end{aligned}$$

Using the facts $a_1 = 0$ and $a_6 = 1$, we obtain

$$\begin{aligned} a_2 &= 0.3a_2 + 0.4a_3 + 0.1, \\ a_3 &= 0.2a_2 + 0.8. \end{aligned}$$

This is a system of two equations in the two unknowns a_2 and a_3, which can be readily solved to yield $a_2 = 21/31$ and $a_3 = 29/31$.

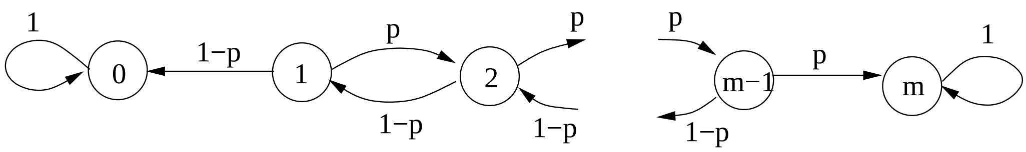
If we want to compute the probabilities of eventually reaching state 1 in the same chain, we will have to use the same equations, but the conditions on a_1 and a_6 will be different. We will have $a_1 = 1$ and $a_6 = 0$, which will yield the equations

$$a_2 = 0.2 + 0.3a_2 + 0.4a_3,$$
$$a_3 = 0.2a_2.$$

The solution of these equations give the desired absorption probabilities $a_2 = 10/31$ and $a_3 = 2/31$.

Example 1.2 (Gambler's ruin problem)**.** A gambler wins \$1 at each round, with probability p, and loses \$1, with probability $1 - p$. Different rounds are assumed independent. The gambler plays continuously until he either accumulates a target amount of \$ m, or loses all his money. What is the probability of eventually accumulating the target amount (winning) or of losing his fortune?

We introduce the Markov chain shown in the next figure whose state i represents the gambler's wealth at the beginning of a round. The states $i = 0$ and $i = m$ correspond to losing and winning, respectively. All states are transient, except for the winning and losing states which are absorbing. Thus, the problem amounts to finding the probabilities of absorption at each one of these two absorbing states. Of course, these absorption probabilities depend on the initial state i.



Let us set $s = 0$ in which case the absorption probability a_i is the probability of losing, starting from state i. These probabilities satisfy

$$a_0 = 1,$$
$$a_i = (1-p)a_{i-1} + pa_{i+1}, \quad i = 1, \ldots, m-1,$$
$$a_m = 0.$$

These equations can be solved in a variety of ways. It turns out there is an elegant method that leads to a nice closed form solution.

Let us write the equations for the a_i as

$$(1-p)(a_{i-1} - a_i) = p(a_i - a_{i+1}), \quad i = 1, \ldots, m-1.$$

Then, by denoting

$$\delta_i = a_i - a_{i+1}, \quad i = 1, \ldots, m-1,$$

and

$$\rho = \frac{1-p}{p},$$

the equations are written as

$$\delta_i = \rho\delta_{i-1}, \quad i = 1, \ldots, m-1,$$

from which we obtain

$$\delta_i = \rho^i \delta_0, \quad i = 1, \dots, m-1.$$

This, together with the equation $\delta_0 + \delta_1 + \cdots + \delta_{m-1} = a_0 - a_m = 1$, implies that $(1 + \rho + \cdots + \rho^{m-1})\delta_0 = 1$. Thus, we have

$$\delta_0 = \begin{cases} \dfrac{1-\rho}{1-\rho^m}, & \rho \neq 1 \\ \dfrac{1}{m}, & \rho = 1. \end{cases}$$

and

$$\delta_i = \begin{cases} \dfrac{\rho^i(1-\rho)}{1-\rho^m}, & \rho \neq 1 \\ \dfrac{1}{m}, & \rho = 1. \end{cases}$$

From this relation, we can calculate the probabilities a_i. If $\rho \neq 1$, we have

$$\begin{aligned} a_i &= a_0 - \delta_{i-1} - \cdots - \delta_0 \\ &= 1 - (\rho^{i-1} + \cdots + \rho + 1)\delta_0 \\ &= 1 - \frac{1-\rho^i}{1-\rho} \cdot \frac{1-\rho}{1-\rho^m} \\ &= 1 - \frac{1-\rho^i}{1-\rho^m} \end{aligned}$$

and therefore the probabilities of losing starting from the fortune i are

$$a_i = \frac{\rho^i - \rho^m}{1-\rho^m}, \quad i = 1, \dots, m-1$$

If $\rho = 1$, we obtain:

$$a_i = \frac{m-i}{i}.$$

The probability of winning starting from fortune i is equal to $1 - a_i$:

$$1 - a_i = \begin{cases} \dfrac{1-\rho^i}{1-\rho^m}, & \rho \neq 1 \\ \dfrac{i}{m}, & \rho = 1. \end{cases}$$

The solution reveals that if $\rho > 1$, which corresponds to $p < 1/2$ and unfavorable odds for the gambler, the probability of losing approaches 1 as $m \to \infty$ regardless of the size of the initial fortune. This suggests that if you aim for a large profit under unfavorable odds, financial ruin is almost certain.