Lecture 30

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April 11, 2005

1 Frequency interpretations

The steady state probabilities π_i's can be viewed as time frequencies of being in certain states. For instance, in the example from previous lecture about the machine, which can be either working or broken, we had:

$$\pi_1 = \frac{r}{b+r}, \quad \pi_2 = \frac{b}{b+r}.$$

That means that π_1 fraction of the time the machine will be working, and π_2 fraction of the time, the machine will be broken.

2 Birth-death processes

Now we will draw our attention to a special case of Markov chains, in which the states are ordered, and the transitions are possible only to the neighboring states. Strictly speaking, we have:

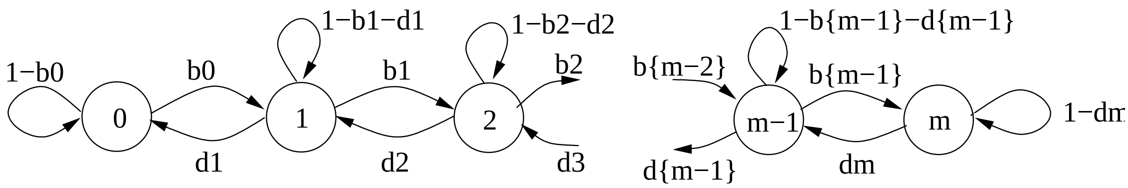
$$b_i = P(X_{n+1} = i+1 | X_n = i) \tag{1}$$

$$d_i = P(X_{n+1} = i-1 | X_n = i) \tag{2}$$

which are called **birth** and **death** probabilities. Moreover, we have, that the probability of staying in the same state is

$$p_{ii} = 1 - b_i - d_i. \tag{3}$$

Graphically, this Markov chain can be represented as follows:



As we see, the probabilities b_i represent the situation, when the number of the state increases (birth), and the probabilities d_i represent the situation when the number of the state decreases (death).

For birth-death Markov chains the balance equations have the following nice form:

$$\pi_i b_i = \pi_{i+1} d_{i+1}, \qquad i = 0, \ldots, m-1. \tag{4}$$

Using these balance equations, we can see that

$$\pi_i = \pi_0 \frac{b_0 b_1 \dots b_{i-1}}{d_1 d_2 \dots d_i}, \quad i = 1, \dots, m. \tag{5}$$

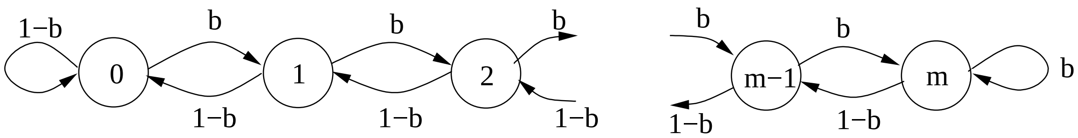
Together with normalization equation

$$\pi_0 + \pi_1 + \cdots + \pi_m = 1 \tag{6}$$

that allows us to find all π_i's.

Example 2.1 (Random Walk with Reflecting Barriers)**.** A person walks along a straight line and, at each time period, takes a step to the right with probability b, and a step to the left with probability $1 - b$. The person starts in one of the positions $1, 2, \dots, m$, but if he reaches position 0 (or position $m + 1$), his step is instantly reflected back to position 1 (or position m, respectively). Equivalently, we may assume that when the person is in positions 1 or m, he will stay in that position with corresponding probability $1 - b$ and b, respectively.

We introduce a Markov chain model whose states are the positions $1, \dots, m$. The transition probability graph of the chain is given below:



The balance equations are

$$\pi_i b = \pi_{i+1}(1 - b), \qquad i = 0, \dots, m - 1.$$

Now, denoting

$$\rho = \frac{b}{1 - b}$$

we have

$$\pi_i = \rho^{i-1} \pi_1, \quad i = 1, \dots, m.$$

Using the normalization equation, we get:

$$\pi_1(1 + \rho + \cdots + \rho^{m-1}) = 1,$$

which leads to

$$\pi_i = \frac{\rho^{i-1}}{1 + \rho + \cdots + \rho^{m-1}}, \quad i = 1, \dots, m$$

Notice that in case $\rho = 1$, we have

$$\pi_i = \frac{1}{m}, \quad i = 1, \dots, m.$$

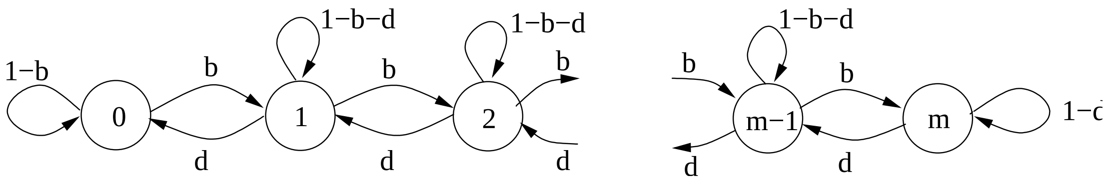
Example 2.2 (Birth-Death Markov Chains Queueing)**.** Packets arrive at a node of a communication network, where they are stored in a buffer and then transmitted. The storage capacity of the buffer is m: if m packets are already present, any newly arriving packets are discarded. We discretize time in very small periods, and we assume that in each period, at most one event can happen that can change the number of packets stored in the node (an arrival of a new packet or a completion of the transmission of an existing packet). In particular, we assume that at each period, exactly one of the following occurs:

(a) one new packet arrives; this happens with a given probability $b > 0$;

(b) one existing packet completes transmission; this happens with a given probability $d > 0$ if there is at least one packet in the node, and with probability 0 otherwise;

(c) no new packet arrives and no existing packet completes transmission; this happens with a probability $1 - b - d$ if there is at least one packet in the node, and with probability $1 - b$ otherwise.

The graph for the problem is given in the next picture:



The balance equations are

$$\pi_i b = \pi_{i+1} d, \qquad i = 0, \ldots, m-1.$$

If we define

$$\rho = \frac{b}{d},$$

we have $\pi_{i+1} = \rho \pi_i$, and $\pi_i = \rho^i \pi_0$ for all i. Using normalization equation $1 = \pi_0 + \cdots + \pi_m$, we have:

$$\pi_0 (1 + \rho + \rho^2 + \cdots + \rho^m) = 1,$$

and therefore,

$$\pi_0 = \begin{cases} \dfrac{1-\rho}{1-\rho^{m+1}}, & \rho \neq 1, \\ \dfrac{1}{m+1}, & \rho = 1. \end{cases}$$

This leads to the following steady state probabilities:

$$\pi_i = \begin{cases} \dfrac{\rho^i (1-\rho)}{1-\rho^{m+1}}, & \rho \neq 1, \\ \dfrac{1}{m+1}, & \rho = 1. \end{cases}$$