Lecture 29

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April 8, 2005

1 Classification of states

In the previous sections we defined Markov chains and started with the state classification. The definition from the previous section was the following:

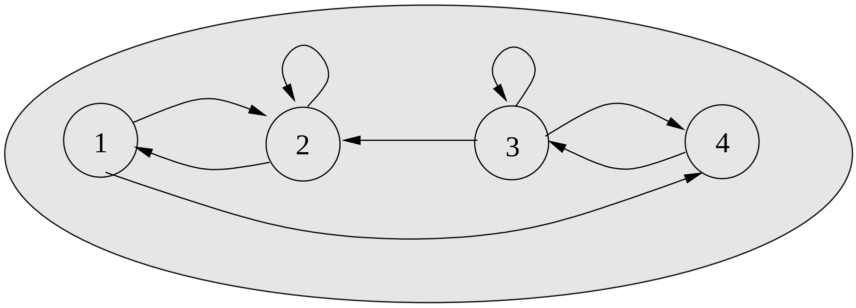
Definition 1.1. *The state i is called* ***recurrent*** *if for all states j, which are accessible from it, i is also accessible from j. All states which are not recurrent are called* ***transient***.

In different words, the state is recurrent is there is always a non-zero probability of returning to it, no matter how the process leaves the state. The state is transient if there is a way to leave this state, such that we will never get back to it again. Please see the example in the previous lecture.

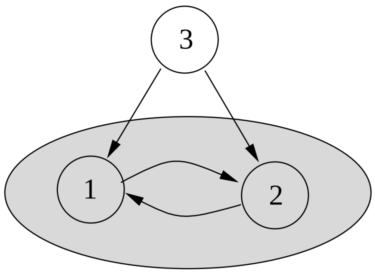
If i is a recurrent state, the set of states $A(i)$ that are accessible from i form a **recurrent class**, meaning that states in $A(i)$ are all accessible from each other, and no state outside $A(i)$ is accessible from them (we never leave the class once we are in it). Moreover, it can be seen that at least one recurrent state must be accessible from any given transient state. This is intuitively evident. It follows that there must exist at least one recurrent state, and hence at least one class. Thus, we reach the following conclusion about the decomposition of the Markov chain:

- A Markov chain can be decomposed into one or more recurrent classes, plus possibly some transient states.
- A recurrent state is accessible from all states in its class, but is not accessible from recurrent states in other classes.
- A transient state is not accessible from any recurrent state.
- At least one, possibly more, recurrent states are accessible from a given transient state.

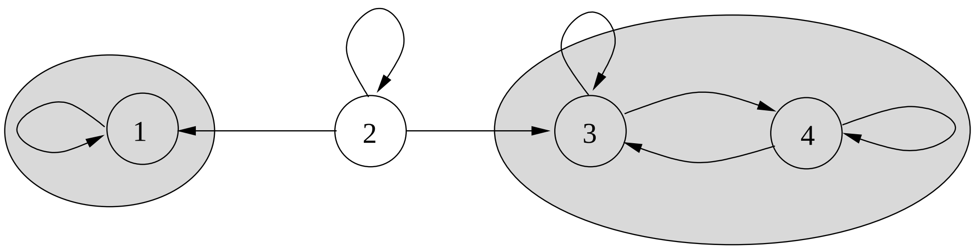
In the following pictures I show the decomposition of the Markov chain into recurrent classes. The classes are shown in grey. On the first picture, the whole chain form one class – from any state we can get to any other state with non-zero probability.



On the second picture states 1 and 2 form a recurrent class – as soon as we are in one of them, we will always be bound to stay in either 1 or 2. State 3 is a transient state – once we leave it, we never get back.

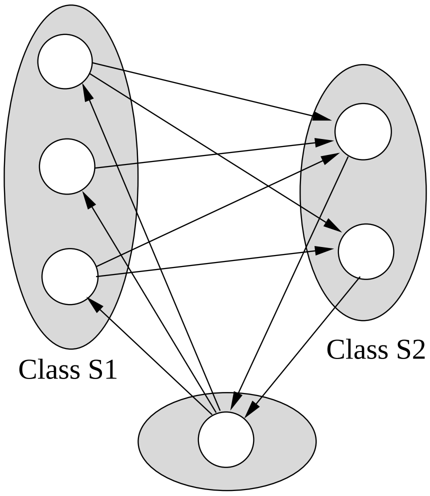


On the next picture, there are two recurrent classes: state 1 is the 1st recurrent class, and the other recurrent class is formed by the state 3 and 4. The state 3 is transient.



2 Periodicity

One more characterization of a recurrent class is of special interest, and relates to the presence or absence of a certain periodic pattern in the times that a state is visited. In particular, a recurrent class is said to be periodic if its states can be grouped in $d > 1$ disjoint subsets $S_1, \ldots, S_d$ so that all transitions from one subset lead to the next subset. The example of the periodic class is given in the next picture. There are 3 classes S_1, S_2, and S_3, and we can see that from the class S_1 we can only get to class S_2, from the class S_2 we can only get to the class S_3, and from the class S_3 we can only get to class S_1.



Recurrent classes without this property are called **aperiodic**.

3 Steady-state behavior

In Markov chain models, we are often interested in long-term state occupancy behavior, that is, in the n-step transition probabilities $r_{ij}(n)$ when n is very large. In the previous examples we saw that the $r_{ij}(n)$ may converge to steady-state values that are independent of the initial state, so to what extent is this behavior typical?

If there are two or more classes of recurrent states, it is clear that the limiting values of the $r_{ij}(n)$ must depend on the initial state (visiting j far into the future will depend on whether j is in the same class as the initial state i). We will, therefore, restrict attention to chains involving a single recurrent class, plus possibly some transient states. Moreover, we will assume that the class is aperiodic.

We now assert that for every state j, the n-step transition probabilities $r_{ij}(n)$ approach a limiting value that is independent of the initial state i. This limiting value, denoted by π_j, has the following interpretation:

$$\pi_j \approx P(X_n = j), \quad \text{when } n \text{ is large}, \tag{1}$$

and is called **steady-state probability** of state j.

The following theorem is the main fact about steady-state probabilities:

Theorem 3.1 (Steady-State Convergence Theorem)**.** *Consider a Markov chain with a single recurrent class, which is aperiodic. Then, the states j are associated with steady-state probabilities π_j that have the following properties.*

(a) $\lim_{n\to\infty} r_{ij}(n) = \pi_j$, *for all i, j.*

(b) The π_j's are the unique solution of the system of equations below:

$$\pi_j = \sum_{k=1}^{m} \pi_k p_{kj}, \quad j = 1, \ldots, m \tag{2}$$

$$1 = \sum_{k=1}^{m} \pi_k. \tag{3}$$

In the matrix form, the first m equations of the system to determine π_j's can be written in the following way:

$$(\pi_1, \pi_2, \ldots, \pi_m) = (\pi_1, \pi_2, \ldots, \pi_m) \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{pmatrix} \tag{4}$$

These equations are called **balance equations**.

Example 3.2. Let's consider the Markov chain with the following transition probability matrix P:

$$P = \begin{pmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{pmatrix}$$

The equations to determine π_1 and π_2 are the following:

$$(\pi_1, \pi_2) = (\pi_1, \pi_2) \begin{pmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{pmatrix}$$
$$1 = \pi_1 + \pi_2,$$

or

$$\pi_1 = 0.8 \cdot \pi_1 + 0.6 \cdot \pi_2$$
$$\pi_2 = 0.2 \cdot \pi_1 + 0.4 \cdot \pi_2$$
$$1 = \pi_1 + \pi_2.$$

Solving this system, we can find that $\pi_1 = 0.75$ and $\pi_2 = 0.25$. Therefore, the probability of being in state 1 after sufficiently large number of steps is equal to 0.75, and the probability of being in state 2 after sufficiently large number of steps is equal to 0.25.

Example 3.3. An absent-minded professor has two umbrellas that she uses when commuting from home to office and back. If it rains and an umbrella is available in her location, she takes it. If it is not raining, she always forgets to take an umbrella. Suppose that it rains with probability p each time she commutes, independently of other times. What is the steady-state probability that she gets wet on a given day?

We model this problem using a Markov chain with the following states:

State i: i umbrellas are available at her current location, $i = 0, 1, 2$.

The transition probability matrix is the following:

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1-p & p \\ 1-p & p & 0 \end{pmatrix}$$

The balance equations for this problem are the following:

$$\pi_0 = (1-p)\pi_2$$
$$\pi_1 = (1-p)\pi_1 + p\pi_2$$
$$\pi_2 = \pi_0 + p\pi_1.$$

From the second equation, we obtain $pi_1 = \pi_2$, which together with the first equation $\pi_0 = (1-p)\pi_2$ and the normalization equation $\pi_0 + \pi_1 + \pi_2 = 1$, yields

$$\pi_0 = \frac{1-p}{3-p}, \quad \pi_1 = \frac{1}{3-p}, \quad \pi_2 = \frac{1}{3-p}.$$

According to the steady-state convergence theorem, the steady-state probability that the professor finds herself in a place without an umbrella is π_0. The steady-state probability that she gets wet is π_0 times the probability of rain p.

Example 3.4. A machine can be either working or broken down on a given day. If it is working, it will break down in the next day with probability b, and will continue working with probability $1-b$. If it breaks down on a given day, it will be repaired and be working in the next day with probability r, and will continue to be broken down with probability $1-r$. What is the steady-state probability that the machine is working on a given day?

We introduce a Markov chain with the following two states:

State 1: Machine is working
State 2: Machine is broken down.

The transition probability matrix is

$$P = \begin{pmatrix} 1-b & b \\ r & 1-r \end{pmatrix}$$

This Markov chain has a single recurrent class that is aperiodic (assuming 0 ¡ b ¡ 1 and 0 ¡ r ¡ 1), and from the balance equations, we obtain

$$\pi_1 = (1-b)\pi_1 + r\pi_2, \quad \pi_2 = b\pi_1 + (1-r)\pi_2,$$

or

$$b\pi_1 = r\pi_2.$$

This equation together with the normalization equation $\pi_1 + \pi_2 = 1$, yields the steady-state probabilities

$$\pi_1 = \frac{r}{b+r}, \quad \pi_2 = \frac{b}{b+r}.$$