Lecture 28

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1 Markov Chains

Last time we were considering the Poisson process – the process according which, for example, the customers arrive at the store. Now we will consider a different type of process.

The time now will be discrete. It means that we have time steps $0, 1, 2, \ldots$. At each of these time instants, the process has some state. Normally, states are numbered from 1 to m, the set of states is denoted by $\mathcal{S} = \{1, 2, \ldots, m\}$. Also, in order to specify Markov chain we have to specify the **transitional probabilities** p_{ij}, which are the probabilities of being in state j on the next step, if now the process is in state i. I.e., if the current moment is n, and X_n denotes the state of the process at step n, we have:

$$p_{ij} = P(X_n = i | X_{n+1} = j), \quad i, j \in \mathcal{S}. \tag{1}$$

From this definition we can notice, that the state, the process enters on the next step depends only on the state at the current moment n, and does not depend on the history of states, visited prior to n. Saying it in different way, for Markov chain the future depends only on the present, and not on the past:

$$P(\underbrace{X_{n+1} = j}_{\text{future}} \,|\, \underbrace{X_n = i}_{\text{present}}, \underbrace{X_{n-1} = i_{n-1}, \ldots, X_1 = i_1, X_0 = i_0}_{\text{past}}) = P(\underbrace{X_{n+1} = j}_{\text{future}} \,|\, \underbrace{X_n = i}_{\text{present}}) = p_{ij}. \tag{2}$$

Obviously, being in some state i at the instant n, we will get to some other state at the instant $n + 1$. That means, that the summation of all transitional probabilities from state i should be equal to 1:

$$\sum_{k=1}^{m} p_{ik} = 1. \tag{3}$$

The probabilities p_{ij} can be arranged in a matrix P, which is called the **matrix of transitional probabilities**:

$$P = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{pmatrix} \tag{4}$$

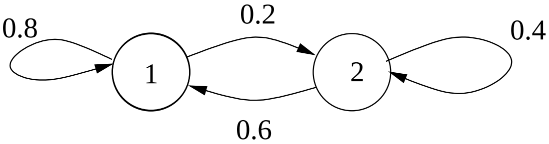
Example 1.1. Let's consider the weather, and assume that the weather has two states: "good" (1) and "bad" (2). Assume that if the weather is good today, the weather will be good (bad) tomorrow with probability 0.8 (0.2). If the weather is bad today, it will be good (bad) tomorrow with probability 0.6 (0.4). The situation described here is clearly a Markov chain process. The weather tomorrow depends only on the weather today, and not on the weather in the past, before today. Here, we have:

$$p_{11} = 0.8, \quad p_{12} = 0.2, \quad p_{21} = 0.6, \quad p_{22} = 0.4,$$

and

$$P = \begin{pmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{pmatrix}$$

Also, this process can be represented by the following graph, where circles correspond to the states, and arrows – to transitions:



Example 1.2. Fly moves along a straight line in unit increments. There is a probability 0.3 that it will move one unit left, 0.3 probability that it will move one unit right, and 0.4 probability that it will stay. The spiders live at the points 1 and m. If the fly gets to the point 1 or m it is eaten by the spider, and the process terminates.

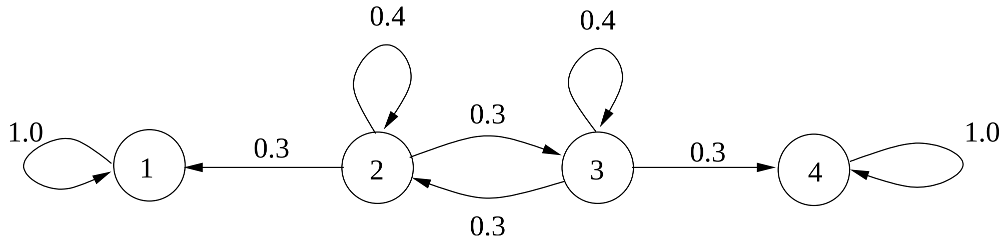
To state this problem as a Markov chain, we will set the transitional probabilities in the following way:

$$p_{11} = 1, \quad p_{mm} = 1, \quad p_{ij} = \begin{cases} 0.3, & \text{if } j = i \pm 1 \\ 0.4, & \text{if } j = i \end{cases}, \text{ for } i = 2, \dots, m-1.$$

In case when $m = 4$, we have the following transition probability matrix:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.3 & 0.4 & 0.3 & 0 \\ 0 & 0.3 & 0.4 & 0.3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The graph, which represents the process, is given on the next figure:



We might be interested in the probability of the sequence of states. The following expression gives what we need:

$$P(X_0 = i_0, X_1 = i_1, \dots, X_n = i_n) = P(X_0 = i_0) p_{i_0 i_1} p_{i_1 i_2} \dots p_{i_{n-1} i_n}, \tag{5}$$

where $P(X_0 = i_0)$ is the probability that at the initial state the process is at the state i_0. Moreover, given the initial state, we have:

$$P(X_1 = i_1, \dots, X_n = i_n | X_0 = i_0) = p_{i_0 i_1} p_{i_1 i_2} \dots p_{i_{n-1} i_n}. \tag{6}$$

In the spider example, we may want to compute the probability of the trajectory of the fly, e.g.

$$P(X_1 = 2, X_2 = 2, X_3 = 3, X_4 = 4 | X_0 = 2) = p_{22} p_{22} p_{23} p_{34} = (0.4)^2 (0.3)^2.$$

2 n-step transition probabilities

As we defined the transition probabilities, we can define the **n-step transition probabilities**, as being the probabilities of being in state j after n steps, if the process initially was is state i:

$$r_{ij}(n) = P(X_n = j | X_0 = i). \tag{7}$$

The regular transitional probabilities are therefore 1-step transitional probabilities:

$$p_{ij} = r_{ij}(1). \tag{8}$$

The n-step transitional probabilities can also be arranged into matrix:

$$P_n = \begin{pmatrix} r_{11}(n) & r_{12}(n) & \dots & r_{1m}(n) \\ r_{21}(n) & r_{22}(n) & \dots & r_{2m}(n) \\ \dots & \dots & \dots & \dots \\ r_{m1}(n) & r_{m2}(n) & \dots & r_{mm}(n) \end{pmatrix} \tag{9}$$

We can compute this matrix by taking n-th power of transition probability matrix:

$$P_n = P^n. \tag{10}$$

Example 2.1. Consider the weather example, as before. We have

$$P = \begin{pmatrix} .8 & .2 \\ .6 & .4 \end{pmatrix}; P^2 = \begin{pmatrix} .76 & .24 \\ .72 & .28 \end{pmatrix}; P^3 = \begin{pmatrix} .752 & .248 \\ .744 & .256 \end{pmatrix}; \dots.$$

If we continue taking powers of this matrix, we will see, that the result approaches the following matrix, which we will denote as P^∞:

$$P^\infty = \begin{pmatrix} .75 & .25 \\ .75 & .25 \end{pmatrix},$$

which will be the matrix of transitional probabilities after infinite number of steps. We can notice, that the rows of this matrix are equal, which means that regardless of the initial state, the probability of being in state 1 after the large number of steps is 0.75, and the probability of being in state 2 after the large number of steps is 0.25.

Example 2.2. Now let's consider the fly-spider example. By multiplying matrix of transitional probabilities by itself sufficiently large number of times, we can see that the result approaches the following matrix:

$$P^\infty = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2/3 & 0 & 0 & 1/3 \\ 1/3 & 0 & 0 & 2/3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This example shows that in this situation after sufficiently large number of steps there is a zero probability of being in states 2 or 3, which means that with probability 1 the fly will be eventually eaten by the spider. Moreover, we can see, that is the fly starts from position 2, it will be eaten by the spider in the position 1 with probability 2/3 and by the spider in the position 4 with probability 1/3, and the opposite hold for the fly, starting in the position 3.

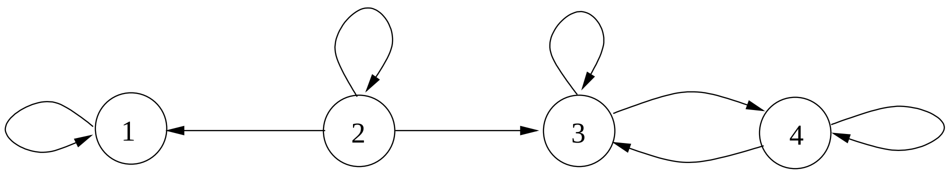
3 Classification of states

In this section we will consider what kind of states can Markov chain have.

Definition 3.1. *The state j is called* ***accessible*** *from state i if for some n, $r_{ij}(n) > 0$, i.e. if there is a non-zero probability of getting to state j from state i after some number of steps.*

For example, on the next graph, we have the following accessibility:

state	states, accessible from it
1	1
2	1,2,3,4
3	3,4
4	3,4



Definition 3.2. *The state i is called* ***recurrent*** *if for all states j, which are accessible from it, i is also accessible from j. All states which are not recurrent are called* ***transient****.*

This definition means, that no matter where we go from state i, there is always a nonzero probability of returning back. On the previous graph, state 2 is not recurrent: if we get out of it either to state 1 or to state 3, we will never get back to state 2 again. Therefore, state 2 is transient. On the contrary, all other states on the graph are recurrent. State 1 is recurrent, because we can not even leave it – once the process got to state 1, it will be returning to it on each next step. State 3 is recurrent, because the process can only get to the state 4 out of it, and there is possibility to return to state 3 after state 4. For the same reasons, the state 4 is recurrent.