Lecture 27

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1 Merging processes

Now we will imaging the situation opposite to the situation we had when we were splitting the process. Imagine the same pharmacy with customers arriving to buy medicines with the rate λ_1 customers per hour, and customers, arriving to buy cigarettes with the rate λ_2 customers per hour. Now, assume we want to characterize the process of all customer's arrivals, regardless, of what they are planning to buy. This process will be a Poisson process with the rate $\lambda_1 + \lambda_2$ customers per hour.

In general, if we have two processes with the rates λ_1 and λ_2, the merged process of all arrivals, regardless of the group they belong to, is a Poisson process with the rate $\lambda_1 + \lambda_2$. Moreover, the probability that the particular arrival is of type I is $\lambda_1/(\lambda_1 + \lambda_2)$, and the probability that the particular arrival is of type II is $\lambda_2/(\lambda_1 + \lambda_2)$.

2 Competing exponents

Assume there are two light bulbs, such that their lifetimes are exponentially distributed with parameters λ_1 and λ_2 respectively. We want to find the first time at which the light bulb burns out. If we denote the lifetimes of light bulbs by T_1 and T_2, we want to find the distribution of the random variable $Z = \min\{T_1, T_2\}$. In order to do it, we will first find the CDF of Z:

$$\begin{aligned} F_Z(z) &= P(\min\{T_1, T_2\} \le z) \\ &= 1 - P(\min\{T_1, T_2\} > z) \\ &= 1 - P(T_1 > z, T_2 > z) \\ &= 1 - P(T_1 > z)P(T_2 > z) \\ &= 1 - e^{-\lambda_1 z} e^{-\lambda_2 z} \\ &= 1 - e^{-(\lambda_1 + \lambda_2) z}. \end{aligned}$$

Therefore, we deduce that $Z \sim Exp(\lambda_1 + \lambda_2)$.

Now, we can use a different approach. Consider the Poisson process associated with the first light bulb. It has rate λ_1. The Poisson process associated with the light bulb of the second type has rate λ_2. We would like to find the distribution of the time of the first arrival in the merged process. The merged process is a Poisson process with the parameter $\lambda_1 + \lambda_2$, and therefore, the time of the first arrival (first time when the light bulb burns out) is distributed exponentially with the parameter $\lambda_1 + \lambda_2$.

Now let's consider 3 light bulbs with identically distributed lifetimes, following the exponential distributions with parameter λ. Now we are interested in the expected time until the last bulb burns out.

We'll look at this problem from the point of view of Poisson processes. Initially, we have 3 light bulbs burning, therefore, we have a Poisson process with the parameter 3λ. The expected time until the first arrival (time until the first light bulb burns out) is equal to $1/3\lambda$. After the first light bulb burnt out, we are left with two light bulbs, and start afresh a new Poisson process with the parameter 2λ. The expected time until the first arrival becomes $1/2\lambda$. Finally, after the second light bulb burns out, we are left with only one light bulb, the expected lifetime of which is $1/\lambda$. Therefore, the expected time until all the light bulbs will burn out is

$$\frac{1}{3\lambda} + \frac{1}{2\lambda} + \frac{1}{\lambda}.$$

3 Incidence paradox

Assume that the buses arrive at the bus stop according to Poisson process with the rate λ. A person comes to the bus stop at some instant of time t and records the length of the interval, during which he/she came. What is the expectation of the values he/she will write?

We might expect, that on average, the person will write the expected interarrival time, which is equal to $1/\lambda$. But this is not true. Assume that the person came at the time t^*, and the previous arrival of the bus was at time U, and the next arrival of the bus is at time V. The person should record $V - U$. What is the expectation of $V - U$? We have:

$$V - U = (t^* - U) + (V - t^*).$$

Because of the fresh start property of the process, $V - t^*$ being the time till the next arrival is exponentially distributed with parameter λ, and therefore $\mathbf{E}[V - t^*] = 1/\lambda$. Now, looking at the process from the time t^* backwards, we see that the expected time since the previous arrival $t^* - U$ is also exponentially distributed with the parameter λ, and therefore $\mathbf{E}[t^* - U] = 1/\lambda$. Thus,

$$\mathbf{E}[V - U] = \mathbf{E}[t^* - U] + \mathbf{E}[V - t^*] = 1/\lambda + 1/\lambda = 2/\lambda,$$

which is two time greater than what we expected.

To explain this paradoxical fact, let's consider the simplified situation, when the buses come deterministically, by the schedule, every hour, and every 15 minutes after the hour, i.e. 9:00am, 9:15am, 10:00am, 10:15am, etc. Therefore, the interarrival times take values 15 or 45 minutes, and the average interarrival time is equal to 30 minutes. The person who comes to the bus stop with probability 3/4 will get there during 45 minute interval, and with the probability 1/4 will get there during 15 minute interval. Therefore, the expected length of the interval, the person will write down, is

$$\frac{3}{4} \cdot 45 + \frac{1}{4} \cdot 15 = 37.5,$$

which is greater than the average time of the interval. The explanation of this fact is pretty simple. Which higher probabilities the person will be getting during longer intervals, so most probably, the person will observe the longer interval, and not the shorter, therefore, the expected value of his observations should be greater than the average.