Lecture 26

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1 Poisson process: Time of the first arrival

Last time we obtained the formula for $P(k, \tau)$ – probability of k arrivals during time τ:

$$P(k, \tau) = e^{-\lambda\tau}\frac{(\lambda\tau)^k}{k!}, \tag{1}$$

which showed that the number of arrivals of the Poisson process during the period of time τ is a Poisson random variable with parameter $\lambda\tau$.

Let's start with an example.

Example 1.1. Assume that the number of accidents during the rush hour from 8am to 9am has rate $\lambda_1 = 5$ accidents per hour, and from 9am to 11am has rate $\lambda_2 = 3$ accidents per hour. What is the distribution of the total number of accidents between 8am and 11am?

The distribution of the number of accidents between 8am and 9am is Poisson(5). The distribution of the number of accidents between 9am and 11am is Poisson$(2 \cdot 3)$ – Poisson(6). Therefore, the total number of accidents between 8am and 11am is a sum of two Poisson random variables with parameters 5 and 6. We know, that sum of two Poisson random variables with parameters λ_1 and λ_2 is a Poisson random variable with parameter $\lambda_1 + \lambda_2$, and therefore the total number of accidents between 8am and 11am is a Poisson random variable with parameter $5 + 6 = 11$.

Another random variable we might consider with respect to Poisson process is the time of the first arrival T. Let's find the distribution of it. First, we will find a CDF of T. We can notice that the fact that the time of the first arrival T is greater than some t is equivalent to the fact that there were no arrivals from 0 to t:

$$\begin{aligned} F_T(t) &= P(T \le t) \\ &= 1 - P(T > t) \\ &= 1 - P(0, t) \\ &= 1 - e^{-\lambda t}. \end{aligned}$$

Therefore, the PDF of the first arrival time is

$$f_T(t) = \lambda e^{-\lambda t}, \tag{2}$$

which shows that the time of the first arrival T is an exponential random variable with parameter λ. Moreover, the expectation and variance of it are

$$\mathbf{E}[T] = \frac{1}{\lambda}; \quad \mathbf{var}(T) = \frac{1}{\lambda^2}. \tag{3}$$

2 Properties of Poisson Process

Now we can look at the properties of the Poisson process.

(a) [**Independence.**] If A and B are non-intersecting time intervals, then arrivals on A are independent of arrivals on B.

(b) [**Fresh-start.**] The portion of the process that starts at any particular time $t > 0$ is independent of the portion of the process before t. We can say, that at any particular instant of time t the process starts afresh.

(c) [**Memorylessness.**] The future and the past of the process are independent. Therefore, the remaining time until the next arrival has the same exponential distribution with parameter λ. To see it, let's concentrate on the moment $t > 0$. Assume that the next arrival will happen at time T. Let's find the distribution of the time till the next arrival $T - t$:

$$\begin{aligned} F_{T-t|T>t}(s) &= P(T - t \leq s|T > t) \\ &= 1 - P(T > t + s|T > t) \\ &= 1 - \frac{P(T > t + s)}{P(T > t)} \\ &= 1 - \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}} \\ &= 1 - e^{-\lambda s}. \end{aligned}$$

Therefore,

$$f_{T-t|T>t}(s) = \lambda e^{-\lambda s}, \tag{4}$$

which means that time till the next arrival at any instant of time t $T - t$ is distributed exponentially with parameter λ.

Example 2.1. You enter the bank and see that there are 3 tellers there, and all of them are busy. Assume that there is no line, so that you go to the first teller which becomes free. What is the probability that you leave the bank last?

Intuition tells us that this probability should be fairly large, since you will be the last person who gets service. Let's see what the theory has to say about it. Let's concentrate on the moment of time when you get to the teller. Because of memorylessness property, the remaining time of service for remaining customers and yours have the same exponential distribution, and therefore, each person has the sam probability of leaving the bank last. Therefore, the probability is 1/3.

3 Interarrival times

The time between arrivals is denoted by T_k (for time between $(k-1)$th and kth arrival), and called **interarrival time**. The distribution of interarrival times is exponential random variable with parameter λ, since if we concentrate on the time of $(k-1)$th arrival, the process starts afresh after that, and the interarrival time is just the time of the first arrival in the newly start process, which is exponential.

Now, we might be interested in the distribution of the time of kth arrival Y_k. We have:

$$Y_k = T_1 + T_2 + \cdots + T_k,$$

so Y_k is a sum of k independent exponential random variables with parameter λ. Therefore, we can see, that

$$\mathbf{E}[Y_k] = \frac{k}{\lambda}, \quad \mathbf{var}(Y_k) = \frac{k}{\lambda^2}. \tag{5}$$

The distribution of the sum of k exponential random variables with parameter λ is called **Erlang ditribution** of order k with parameter λ, and the formula for its PDF is as follows:

$$f_{Y_k}(y) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}. \tag{6}$$

4 Splitting process

Let's consider the customers arriving at the pharmacy with a certain rate of λ customers per hour. Assume, that with the probability p a customer is buying medicines, and with probability $1-p$ – cigarettes. Then, the process of arrivals of customers, who came only to buy medicines will be a Poisson process with the rate λp.

Generally speaking, if each arrival of the Poisson process is kept with the probability p and discarded with probability $1-p$, the remaining arrivals form a Poisson process with the parameter λp.