Lecture 21

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1 Sum of random number of random variables

In this lecture we will continue studying the sum of a random number of random variables.

Example 1.1 (Sum of Geometric Number of Exponential RVs). Assume George wants to find the book. Each bookstore carries it with probability p, independent of the others. George visits different bookstores, until he finds it, and in each bookstore he spends time, which is exponentially distributed with parameter λ. What is the average time till he finds the book?

The number of the store in which George finds the book N is a geometric random variable ("success" – is to find the book, "failure" – no book in the store, probability of "success" is p). The total time till he find the book is a summation of geometric number N of exponentially distributed times X_i:

$$Y = X_1 + X_2 + \cdots + X_N.$$

We have:

$$\mathbf{E}[N] = \frac{1}{p}, \qquad \mathbf{var}(N) = \frac{1-p}{p^2}$$
$$\mathbf{E}[X_i] = \frac{1}{\lambda}, \qquad \mathbf{var}(X_i) = \frac{1}{\lambda^2}.$$

Now, using the formulae obtained before, we have:

$$\mathbf{E}[Y] = \mathbf{E}[N]\,\mathbf{E}[X_i] = \frac{1}{p}\cdot\frac{1}{\lambda} = \frac{1}{p\lambda}$$
$$\mathbf{var}(Y) = \mathbf{E}[N]\,\mathbf{var}(X_i) + (\mathbf{E}[X_i])^2\mathbf{var}(N) = \frac{1}{p}\cdot\frac{1}{\lambda^2} + \frac{1}{\lambda^2}\cdot\frac{1-p}{p^2} = \frac{1}{\lambda^2 p^2}.$$

Now let's look at the transform of the sum of random number of random variables. Again, assuming that $X_1, X_2, \ldots$ and N are independent random variables, and that $X_1, X_2, \ldots$ are identically distributed, for $Y = X_1 + \cdots + X_N$ we have:

$$\begin{aligned}
\mathbf{E}\left[e^{sY}|N=n\right] &= \mathbf{E}\left[e^{s(X_1+X_2+\cdots+X_N)}|N=n\right] \\
&= \mathbf{E}\left[e^{sX_1} + e^{sX_2} + \cdots + e^{sX_N}|N=n\right] \\
&= \mathbf{E}\left[e^{sX_1}e^{sX_2}\ldots e^{sX_n}\right] \\
&= \mathbf{E}\left[e^{sX_1}\right]\mathbf{E}\left[e^{sX_2}\right]\ldots\mathbf{E}\left[e^{sX_n}\right] \\
&= (M_X(s))^n.
\end{aligned}$$

Now, using the law of iterated expectations, we have:

$$\begin{aligned}
M_Y(s) &= \mathbf{E}\left[e^{sY}\right] \\
&= \mathbf{E}\left[\mathbf{E}\left[e^{sY}|N=n\right]\right] \\
&= \mathbf{E}\left[(M_X(s))^n\right] \\
&= \sum_{n=0}^{\infty}(M_X(s))^n p_N(n),
\end{aligned}$$

Which is the same sum as in the definition of $M_N(n)$:

$$M_N(n) = \sum_{n=0}^{\infty}(e^s)^n p_N(n),$$

except that instead of e^s we now have $M_X(s)$. That suggests us the algorithm of finding the transform of Y: We need to take the transform of N, and instead of e^s substitute there $M_X(s)$.

Example 1.2. Continuing with the bookstore example, where we had a geometric number of exponentially distributed random variables, we have:

$$M_X(s) = \frac{\lambda}{\lambda - s}, \qquad M_N(s) = \frac{pe^s}{1-(1-p)e^s}$$

$$M_Y(s) = \frac{p\frac{\lambda}{\lambda-s}}{1-(1-p)\frac{\lambda}{\lambda-s}} = \frac{p\lambda}{p\lambda - s},$$

which is a characteristic function of exponential random variable with parameter $p\lambda$. It is interesting to compare it to the example from the previous lecture, where we figured out that the sum of fixed number of exponential random variables was not exponential, but here, sum of geometric number of exponential random variables is still an exponential random variable.

Example 1.3. Now let's consider the sum of geometric number of geometric random variables. If all X_i's are geometric with parameter q, and N is geometric with parameter p, we have:

$$M_X(s) = \frac{qe^s}{1-(1-q)e^s}, \qquad M_N(s) = \frac{pe^s}{1-(1-p)e^s}$$

$$M_Y(s) = \frac{pM_X(s)}{1-(1-p)M_X(s)} = \frac{pqe^s}{1-(1-pq)e^s},$$

from where we can conclude that sum of geometric number of geometric random variables is a geometric random variables with parameter pq.

2 Covariance

Definition 2.1. ***Covariance*** *of two random variables X and Y is*

$$\mathrm{Cov}(X,Y) = \mathbf{E}\left[(X-\mathbf{E}[X])(Y-\mathbf{E}[Y])\right]. \tag{1}$$

The covariance is a measure of dependency between two random variables. It means, that if covariance $\mathrm{Cov}(X, Y)$ is positive, then with the increase of X, Y also increases (on average). If $\mathrm{Cov}(X, Y)$ is negative, then as X increases, Y decreases. Two random variables X and Y such that $\mathrm{Cov}(X, Y) = 0$ are called **uncorrelated**.

Now let's get another formula for covariance:

$$\begin{aligned}
\mathrm{Cov}(X, Y) &= \mathbf{E}\left[(X - \mathbf{E}[X])(Y - \mathbf{E}[Y])\right] \\
&= \mathbf{E}\left[XY - \mathbf{E}[X] \cdot Y - X \cdot \mathbf{E}[Y] + \mathbf{E}[X]\,\mathbf{E}[Y]\right] \\
&= \mathbf{E}[XY] - \mathbf{E}[X]\,\mathbf{E}[Y] - \mathbf{E}[X]\,\mathbf{E}[Y] + \mathbf{E}[X]\,\mathbf{E}[Y] \\
&= \mathbf{E}[XY] - \mathbf{E}[X]\,\mathbf{E}[Y].
\end{aligned}$$

This formula is sometimes easier to use than the original definition of covariance.

What is the covariance of X with itself?

$$\begin{aligned}
\mathrm{Cov}(X, X) &= \mathbf{E}\left[X^2\right] - \mathbf{E}[X]\,\mathbf{E}[X] \\
&= \mathbf{var}(X).
\end{aligned}$$

The important fact about covariance, is that if X and Y are independent random variables, then $\mathrm{Cov}(X, Y) = 0$:

$$\begin{aligned}
\mathrm{Cov}(X, Y) &= \mathbf{E}[XY] - \mathbf{E}[X]\,\mathbf{E}[Y] \\
&= \mathbf{E}[X]\,\mathbf{E}[Y] - \mathbf{E}[X]\,\mathbf{E}[Y] \qquad (X, Y \text{ independent } \Rightarrow \mathbf{E}[XY] = \mathbf{E}[X]\,\mathbf{E}[Y]) \\
&= 0.
\end{aligned}$$

The converse of this fact is not true.

Example 2.2. Assume X and Y are random variable, such that the only pairs of values they can take are $(1, 0), (0, 1), (-1, 0), (0, -1)$, and each of these pair have the same probability $1/4$. In this case obviously X and Y are dependent, since, for example, knowing that $X \neq 0$, we can deduce that $Y = 0$. Now,

$$p_X(x) = \begin{cases} 1/4, & x = -1 \\ 1/2, & x = 0 \\ 1/4, & x = 1 \end{cases} \qquad p_Y(y) = \begin{cases} 1/4, & y = -1 \\ 1/2, & y = 0 \\ 1/4, & y = 1 \end{cases}$$

Therefore,

$$\mathbf{E}[X] = 0, \qquad \mathbf{E}[Y] = 0.$$

Moreover, since one of the numbers in the pair is always equal to 0, $XY = 0$, and therefore, $\mathbf{E}[XY] = 0$. Using the formula for covariance, we get:

$$\mathrm{Cov}(X, Y) = \mathbf{E}[XY] - \mathbf{E}[X]\,\mathbf{E}[Y] = 0.$$

So, random variables in this example are dependent, though their covariance is equal to 0.