# Lecture 19

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March 09, 2005

## 1 Transform of the sum of two independent RVs

Assume $X$ and $Y$ are independent random variables with transforms $M_X(s)$ and $M_Y(s)$ respectively. Let $W = X + Y$. Let' find the transform of $W$:

$$\begin{aligned} M_W(s) &= \mathbf{E}\left[e^{sW}\right] \\ &= \mathbf{E}\left[e^{s(X+Y)}\right] \\ &= \mathbf{E}\left[e^{sX+sY}\right] \\ &= \mathbf{E}\left[e^{sX}e^{sY}\right] \\ &= \mathbf{E}\left[e^{sX}\right]\mathbf{E}\left[e^{sY}\right] && \text{since } X \text{ and } Y \text{ are independent} \\ &= M_X(s)M_Y(s). \end{aligned}$$

So, we have:

$$M_W(s) = M_X(s)M_Y(s), \qquad W = X + Y. \tag{1}$$

The following examples will demonstrate the use of this formula.

**Example 1.1 (Binomial Distribution).** First, let's find the transform of the Bernoulli random variable. If $Y$ is a Bernoulli random variable, which takes value 1 with probability $p$ and value 0 with probability $1 - p$, it's transform is

$$M_Y(s) = (1-p)e^{0s} + pe^{1s} = 1 - p + pe^s.$$

Now, let $X$ be a binomial random variable with parameters $(n,p)$. We might recall, that the binomial random variable can be represented as a sum of $n$ Bernoulli random variables. Therefore, using the formula (1) we get:

$$M_X(s) = (1 - p + pe^s)^n. \tag{2}$$

**Example 1.2 (Sum of Poisson Random Variables).** Let $X$ and $Y$ be Poisson random variables with parameters $\lambda$ and $\mu$ respectively. Then,

$$M_X(s) = e^{\lambda(e^s-1)}, \qquad M_Y(s) = e^{\mu(e^s-1)}.$$

If $W = X + Y$, we have:

$$\begin{aligned} M_W(s) &= M_X(s)M_Y(s) \\ &= e^{\lambda(e^s-1)} \times e^{\mu(e^s-1)} \\ &= e^{\lambda(e^s-1)\times\mu(e^s-1)} \\ &= e^{(\lambda+\mu)(e^s-1)}. \end{aligned}$$

The last expression is a transform of a Poisson random variable with parameter $\lambda+\mu$. Therefore, $W$ is a Poisson random variable with parameter $\lambda + \mu$.

**Example 1.3 (Sum of Normal Random Variables).** Let $X$ and $Y$ be normal random variables with parameters $(\mu_X, \sigma_X^2)$ and $(\mu_Y, \sigma_Y^2)$ respectively. Then we have:

$$M_X(s) = \exp\left\{\frac{s^2\sigma_X^2}{2} + \mu_X s\right\}, \qquad M_Y(s) = \exp\left\{\frac{s^2\sigma_Y^2}{2} + \mu_Y s\right\}.$$

If $W = X + Y$, we have:

$$\begin{aligned} M_W(s) &= M_X(s)M_Y(s) \\ &= \exp\left\{\frac{s^2\sigma_X^2}{2} + \mu_X s\right\} \times \exp\left\{\frac{s^2\sigma_Y^2}{2} + \mu_Y s\right\} \\ &= \exp\left\{\frac{s^2\sigma_X^2}{2} + \mu_X s + \frac{s^2\sigma_Y^2}{2} + \mu_Y s\right\} \\ &= \exp\left\{\frac{s^2(\sigma_X^2 + \sigma_Y^2)}{2} + (\mu_X + \mu_Y)s\right\}, \end{aligned}$$

which is a transform of the normal distribution with parameters $(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$:

$$X \sim N(\mu_X, \sigma_X^2), Y \sim N(\mu_Y, \sigma_Y^2) \Rightarrow W = X + Y \sim N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2).$$

**Example 1.4.** Now let's see if this approach works for the case of two exponential distributions. Let $X$ and $Y$ be exponential random variables with parameters $\lambda$ and $\mu$ respectively. In this case

$$M_X(s) = \frac{\lambda}{\lambda - s}, \qquad M_Y(s) = \frac{\mu}{\mu - s}.$$

If $W = X + Y$, we have

$$M_W(s) = M_X(s)M_Y(s) = \frac{\lambda}{\lambda - s}\frac{\mu}{\mu - s} = \frac{\lambda\mu}{(\lambda - s)(\mu - s)}.$$

This is not a transform of exponential distribution, therefore, we can deduce that sum of two exponential random variables is not an exponential random variable. Moreover, since we've never seen this transform before, we can not even identify this distribution.

If $X_1, X_2, \ldots, X_n$ are random variables, it is also possible to define their joint transform as

$$M_{X_1,X_2,\ldots,X_n}(s_1, s_2, \ldots, s_n) = \mathbf{E}\left[e^{s_1X_2+s_2X_2+\cdots+s_nX_n}\right]. \tag{3}$$

# 2 Conditional Expectations as a Random Variable

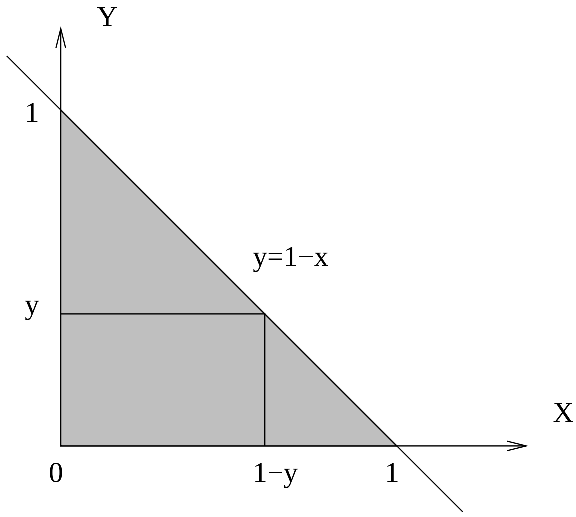
If $X$ and $Y$ are two random variables, we can consider the conditional expectation of $X$ given that $Y = y$: $\mathbf{E}[X|Y = y]$. In general, this conditional expectation of $X$ depends on $Y$. Therefore, $\mathbf{E}[X|Y]$ is a function of $Y$, and therefore it is a random variable. Let me recall the definitions:

$$\mathbf{E}[X|Y = y] = \sum_x x p_{X|Y}(x|y), \qquad X \text{ is discrete,} \tag{4}$$

and

$$\mathbf{E}[X|Y = y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\, dx, \qquad X \text{ is continuous.} \tag{5}$$

**Example 2.1.** Let $X$ and $Y$ be uniformly distributed in the shaded region from the next figure ($X \geq 0$, $Y \geq 0$, $X + Y \leq 1$):




The area of this region is equal to 1/2, and therefore

$$f_{X,Y}(x, y) = 2, \qquad (x, y) \text{ in the region.}$$

We can calculate the marginal PDF of $Y$:

$$f_Y(y) = \int_0^{1-y} f_{X,Y}(x, y)\, dx = \int_0^{1-y} 2\, dx = 2(1 - y), \quad 0 \leq y \leq 1.$$

Now, we can compute the conditional PDF of $X$:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x, y)}{f_Y(y)} = \frac{2}{2(1 - y)} = \frac{1}{1 - y}, \quad 0 \leq x \leq 1 - y.$$

Therefore, we can see that given $Y = y$, $X$ is a uniform random variable, distributed from 0 to $1 - y$. Thus, we have:

$$\mathbf{E}[X|Y = y] = \frac{1 - y}{2},$$

and therefore,

$$\mathbf{E}[X|Y] = \frac{1 - Y}{2}.$$

Let's find the expectation of $\mathbf{E}[X|Y]$. We have:

$$\mathbf{E}[\mathbf{E}[X|Y]] = \begin{cases} \sum_y \mathbf{E}[X|Y = y]\, p_Y(y), & Y \text{ is discrete} \\ \int_{-\infty}^{\infty} \mathbf{E}[X|Y = y]\, f_Y(y)\, dy, & Y \text{ is continuous} \end{cases}$$

From the total expectation theorem, we have the following important equality, which is called the **law of iterated expectations:**

$$\mathbf{E}[\mathbf{E}[X|Y]] = \mathbf{E}[X]. \tag{6}$$

**Example 2.2.** Let's continue with the previous example. From the law of iterated expectations and the expression for $\mathbf{E}[X|Y]$ we have:

$$\mathbf{E}[X] = \mathbf{E}[\mathbf{E}[X|Y]] = \mathbf{E}\left[\frac{1-Y}{2}\right] = \frac{1-\mathbf{E}[Y]}{2}.$$

Since $X$ and $Y$ are distributed symmetrically, we have

$$\mathbf{E}[X] = \mathbf{E}[Y],$$

and

$$\mathbf{E}[X] = \frac{1-\mathbf{E}[X]}{2},$$

from where $\mathbf{E}[X] = 1/3$.

**Example 2.3.** Assume there is a stick of length $l$, which you break at a random point of it, and keep the left part. Then you break it again, and again keep the left part. What is the expected length of what is left?

Let $Y$ be the length of the remaining part after the first break. Let $X$ be the length of the remaining part after the second break. We have:

$$\mathbf{E}[Y] = \frac{l}{2},$$

since the place of the first break is uniformly distributed from 0 to $l$. Now, we have a piece of stick of length $Y$, and we break it at some random point. The break location is uniformly distributed from 0 to $Y$, and therefore,

$$\mathbf{E}[X|Y] = \frac{Y}{2}.$$

Now, we have:

$$\mathbf{E}[X] = \mathbf{E}[\mathbf{E}[X|Y]] = \mathbf{E}\left[\frac{Y}{2}\right] = \frac{\mathbf{E}[Y]}{2} = \frac{l}{4}.$$