Lecture 18

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1 Transforms

Now we will compute transforms of several distributions. As a reminder, the definition of the transform is given below:

$$M_X(s) = \mathbf{E}\left[e^{sX}\right] = \begin{cases} \sum_x e^{sx} p_X(x), & X \text{ is discrete} \\ \int_{-\infty}^{\infty} e^{sx} f_X(x)\,dx, & X \text{ is continuous} \end{cases} \tag{1}$$

Example 1.1 (Poisson Distribution). The PMF of Poisson distribution is:

$$p_X(x) = e^{-\lambda} \cdot \frac{\lambda^x}{x!}, \qquad x = 0, 1, \ldots$$

Now,

$$\begin{aligned} M_X(s) &= \sum_{x=0}^{\infty} e^{sx} \frac{\lambda^x e^{-\lambda}}{x!} \\ &= \sum_{x=0}^{\infty} \frac{(e^s\lambda)^x e^{-\lambda}}{x!} \\ &= e^{-\lambda} \sum_{x=0}^{\infty} \frac{a^x}{x!} && \text{substitute } a = e^s\lambda \\ &= e^{-\lambda} e^a && \sum_{x=0}^{\infty} \frac{a^x}{x!} \text{ is a Taylor expansion of } e^a \\ &= e^{-\lambda} e^{e^s\lambda} \\ &= e^{\lambda(e^s-1)}. \end{aligned}$$

Therefore, for Poisson distribution,

$$M_X(s) = e^{\lambda(e^s-1)} \tag{2}$$

Example 1.2 (Exponential Distribution). The PDF ofexponential distribution is

$$f_X(x) = \lambda e^{-\lambda x}, \qquad x \geq 0.$$

Now,

$$\begin{aligned}
M_X(s) &= \lambda \int_0^\infty e^{-\lambda x} e^{sx}\, dx \\
&= \lambda \int_0^\infty e^{x(s-\lambda)}\, dx \\
&= \frac{\lambda}{s-\lambda}\, e^{(s-\lambda)x}\Big|_{x=0}^{\infty} \\
&= \frac{\lambda}{\lambda - s} && \text{considering only the case when } s < \lambda.
\end{aligned}$$

Therefore, for exponential distribution,

$$M_X(s) = \frac{\lambda}{\lambda - s}, \qquad s < \lambda. \tag{3}$$

Now let's see what is the transform of the linear function of the random variable. Assume, X is a random variable, and transform of X is $M_X(s)$. Let $Y = aX + b$. Now,

$$\begin{aligned}
M_Y(s) &= \mathbf{E}\left[e^{sY}\right] \\
&= \mathbf{E}\left[e^{s(aX+b)}\right] \\
&= \mathbf{E}\left[e^{sb}e^{saX}\right] \\
&= e^{sb}\mathbf{E}\left[e^{saX}\right] && \text{since } e^{sb} \text{ is a constant} \\
&= e^{sb} M_X(sa).
\end{aligned}$$

Therefore, we have:

$$M_Y(s) = e^{sb} M_X(sa), \qquad Y = aX + b \tag{4}$$

Example 1.3. Let X be an exponential random variable with parameter 2. Therefore, from the example (1.2),

$$M_X(s) = \frac{2}{2-s}.$$

Now, let $Y = 3X + 5$. From (4), we have:

$$M_Y(y) = e^{5s} \frac{2}{2-3s}$$

Example 1.4 (Normal Distribution). First, we will compute the transform of standard normal distribution. Let Y be a standard normal random variable, such that

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}.$$

We have:

$$
\begin{aligned}
M_Y(s) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} e^{sy}\, dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}+sy}\, dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}+sy-\frac{s^2}{2}+\frac{s^2}{2}}\, dy \\
&= \frac{1}{\sqrt{2\pi}} e^{\frac{s^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(y^2-2sy+s^2)}\, dy \\
&= e^{\frac{s^2}{2}} \times \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(y-s)^2}\, dy \\
&= e^{\frac{s^2}{2}}
\end{aligned}
$$

where,

$$
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(y-s)^2}\, dy = 1
$$

because it's an integral of the density (PDF) of normal distribution with mean s and variance 1 $N(s,1)$.

Now, let X be a normal random variable with mean μ and variance σ^2:

$$
X \sim N(\mu, \sigma^2).
$$

In this case, we have:

$$
X = \sigma Y + \mu,
$$

and therefore, using the formula (4), we have:

$$
M_X(s) = e^{s\mu} M_Y(s\sigma) = e^{\frac{\sigma^2 s^2}{2}+\mu s} \tag{5}
$$

2 Moments and transforms

Let me first remind the definition of the moment. **n-th moment** of random variable X is $\mathbf{E}[X^n]$. Let's explore the connection between transforms and moments. We have:

$$
M_X(s) = \int_{-\infty}^{\infty} e^{sx} f_X(x)\, dx.
$$

Differentiating both parts of this equality with respect to s, we have:

$$
\begin{aligned}
\frac{d}{ds} M_X(s) &= \frac{d}{ds} \int_{-\infty}^{\infty} e^{sx} f_X(x)\, dx \\
&= \int_{-\infty}^{\infty} \frac{d}{ds} e^{sx} f_X(x)\, dx \\
&= \int_{-\infty}^{\infty} x e^{sx} f_X(x)\, dx.
\end{aligned}
$$

Now, let's substitute $s = 0$:

$$\frac{d}{ds} M_X(s)\bigg|_{s=0} = \int_{-\infty}^{\infty} x f_X(x)\, dx = \mathbf{E}\left[X\right].$$

Therefore, we obtained the following formula:

$$M_X'(0) = \mathbf{E}\left[X\right]. \tag{6}$$

Generally, the following equality is also true:

$$\frac{d^n}{ds^n} M_X(s)\bigg|_{s=0} = \mathbf{E}\left[X^n\right]. \tag{7}$$

3 Inversion property

One of the main fact about the transforms is the following: **transform $M_X(s)$ completely determines the distribution of the random variable X.** It means that if transforms of two random variable X and Y are equal:

$$M_X(s) = M_Y(s) \quad \forall s,$$

then X and Y have the same probability distribution.

Now, let me repeat the example from the previous lecture.

Example 3.1. Let X be a discrete random variable with the following PMF:

$$p_X(x) = \begin{cases} 1/2, & x = 2 \\ 1/6, & x = 3 \\ 1/3, & x = 5 \\ 0, & \text{otherwise} \end{cases}$$

In this case, we have:

$$\begin{aligned} M_X(s) &= \mathbf{E}\left[e^{sX}\right] \\ &= \sum_x e^{sx} p_X(x) \\ &= \frac{1}{2} \cdot e^{2s} + \frac{1}{6} \cdot e^{3s} + \frac{1}{3} \cdot e^{5s}. \end{aligned}$$

From this example we can notice, that the degrees of exponents are equal to possible values of X, and the coefficients before the exponents are equal to the probabilities.

Now assume, that you are told that the transform of the random variable is

$$M_X(s) = \frac{1}{4} \cdot e^{-s} + \frac{1}{2} + \frac{1}{8} \cdot e^{4s} + \frac{1}{8} \cdot e^{5s}.$$

This will allow you to identify the distribution of X. Since 1 is equal to e^{0s}, we can see that the coefficient $\frac{1}{2}$ in the transform is a probability of $X = 0$. Therefore, we deduce, that

$$p_X(x) = \begin{cases} 1/4, & x = -1 \\ 1/2, & x = 0 \\ 1/8, & x = 4 \\ 1/8, & x = 5 \\ 0, & \text{otherwise} \end{cases}$$

In the next example we will compute the transform of the geometric distribution.

Example 3.2 (Geometric Distribution). The PMF of the geometric random variable X is

$$p_X(x) = p(1-p)^{x-1}, \qquad x = 1, 2, \ldots$$

Now,

$$\begin{aligned}
M_X(s) &= \sum_{x=1}^{\infty} e^{sx} p(1-p)^{x-1} \\
&= e^s p \times \sum_{x=1}^{\infty} e^{s(x-1)} (1-p)^{x-1} \\
&= e^s p \times \sum_{y=0}^{\infty} e^{sy} (1-p)^{y} \\
&= e^s p \times \sum_{y=0}^{\infty} [(1-p)e^s]^{y} \\
&= e^s p \frac{1}{1-(1-p)e^s} && \text{using the formula } 1 + a + a^2 + \cdots = \frac{1}{1-a}.
\end{aligned}$$

Therefore, the transform of the geometric random variable is

$$M_X(s) = \frac{pe^s}{1-(1-p)e^s} \tag{8}$$