Lecture 17

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1 Convolutions

In this section we will consider the situation in which we want to find the distribution of the sum of two random variables.

First let's consider the discrete case. Assume X and Y are independent discrete random variables with PMFs $p_X(x)$ and $p_Y(y)$ respectively. Assume that the random variable W is a sum of X and Y:

$$W = X + Y.$$

If we want to find the probability that W is equal to w. In this case we have take all possible values of X and Y that add up to w, and add up all the probabilities. Formalizing, we have the following:

$$\begin{aligned} P_W(w) &= P(W = w) \\ &= P(X + Y = w) \\ &= \sum_{x,y:x+y=w} P(X = x, Y = y) \\ &= \sum_x P(X = x, Y = w - x) \\ &= \sum_x P(X = x)P(Y = w - x) \\ &= \sum_x p_X(x)p_Y(w - x). \end{aligned}$$

Therefore, the formula we obtained is the following:

$$P_W(w) = \sum_x p_X(x)p_Y(w - x). \tag{1}$$

The expression in the right hand side of the formula is called **(discrete) convolution of functions $p_X(x)$ and $p_Y(y)$**.

Example 1.1. Let X and Y be the random variables with the following PMFs:

$$p_X(x) = \begin{cases} 1/3, & x = 1, 2, 3 \\ 0, & \text{otherwise} \end{cases} \qquad p_Y(y) = \begin{cases} 1/2, & y = 0 \\ 1/3, & y = 1 \\ 1/6, & y = 2 \\ 0, & \text{otherwise} \end{cases}$$

Let $W = X + Y$. The possible values for W are 1,2,3,4,5. Now we will find the PMF of W.

$$\begin{aligned}
p_W(1) &= p_X(1)p_Y(0) = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6} \\
p_W(2) &= p_X(1)p_Y(1) + p_X(2)p_Y(0) = \frac{1}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{1}{2} = \frac{5}{18} \\
p_W(3) &= p_X(1)p_Y(2) + p_X(2)p_Y(1) + p_X(3)p_Y(0) = \frac{1}{3} \times \frac{1}{6} + \frac{1}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{1}{2} = \frac{1}{3} \\
p_W(4) &= p_X(2)p_Y(2) + p_X(3)p_Y(1) = \frac{1}{3} \times \frac{1}{6} + \frac{1}{3} \times \frac{1}{3} = \frac{1}{6} \\
p_W(5) &= p_X(3)p_Y(2) = \frac{1}{3} \times \frac{1}{6} = \frac{1}{18}.
\end{aligned}$$

Now we'll move the the continuous case. Assume X and Y are independent random variables with PDFs $f_X(x)$ and $f_Y(y)$ respectively. Let $W = X + Y$. We will first look for CDF of W:

$$\begin{aligned}
F_W(w) &= P(W \leq w) \\
&= P(X + Y \leq w) \\
&= \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{w-x} f_X(x) f_Y(y)\, dy\, dx \\
&= \int_{x=-\infty}^{\infty} f_X(x) \left[\int_{y=-\infty}^{w-x} f_Y(y)\, dy \right] dx \\
&= \int_{x=-\infty}^{\infty} f_X(x) F_Y(w - x)\, dx.
\end{aligned}$$

Now differentiating left-hand and right-hand sides of the equality with respect to w, we get:

$$f_W(w) = \int_{x=-\infty}^{\infty} f_X(x) f_Y(w - x)\, dx. \tag{2}$$

This expression is called the **convolution of functions $f_X(x)$ and $f_Y(y)$**.

Example 1.2. Let X and Y be uniform random variables on $[0, 1]$, so that their respective PDFs are:

$$f_X(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases} \qquad f_Y(y) = \begin{cases} 1, & 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

We will find the distribution of their sum $W = X + Y$ using the convolution formula. The integrand is equal to

$$f_X(x) f_Y(w - x).$$

The first part of the formula is equal to zero everywhere except for

$$0 \leq x \leq 1.$$

The second part is equal to zero everywhere except for

$$\begin{aligned}
&0 \leq w - x \leq 1 \\
&-w \leq -x \leq 1 - w \\
&w \geq x \geq w - 1 \\
&w - 1 \leq x \leq w.
\end{aligned}$$

Therefore, we got two conditions on x, which I will summarize here:

$$0 \leq x \leq 1$$
$$w - 1 \leq x \leq w.$$

To intersect these two intervals, we should take maximum of their right ends, and minimum of their left ends. Thus, the final condition on X is

$$\max\{0, w-1\} \leq x \leq \min\{1, w\}.$$

This is the interval, on which the integrand is equal to 1. Outside of it, integrand is equal to 0. Therefore, by the convolution formula we get:

$$f_W(w) = \int_{\max\{0,w-1\}}^{\min\{1,w\}} 1\, dx = \min\{1, w\} - \max\{0, w-1\}$$

In order to visualize this function, we will consider two cases: when $0 \leq w \leq 1$ and when $1 \leq w \leq 2$.

(i) $0 \leq w \leq 1$: In this case,

$$\min\{1, w\} = w$$
$$\max\{0, w-1\} = 0,$$

and therefore

$$f_W(w) = w, \qquad 0 \leq w \leq 1.$$

(ii) $1 \leq w \leq 2$: In this case,

$$\min\{1, w\} = 1$$
$$\max\{0, w-1\} = w - 1,$$

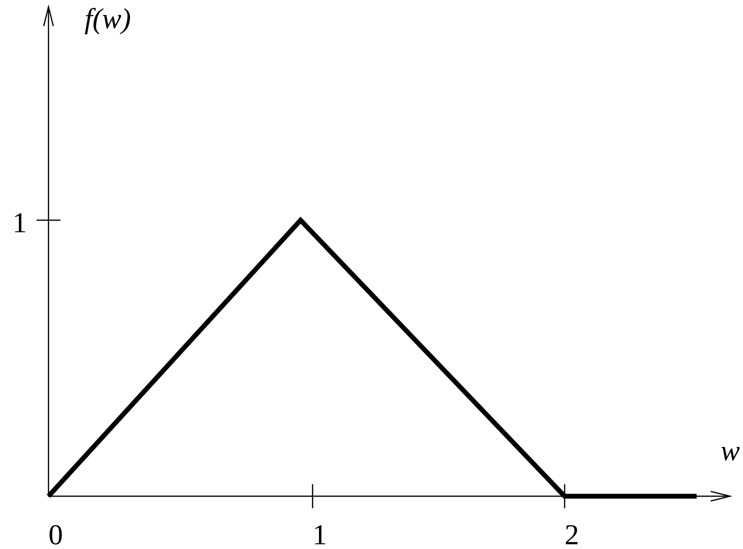
and therefore

$$f_W(w) = 1 - (w - 1) = 2 - w, \qquad 1 \leq w \leq 2.$$

Therefore, the distribution of W has the following PDF:

$$f_W(w) = \begin{cases} w, & 0 \leq w \leq 1 \\ 1 - 2w, & 1 \leq w \leq 2. \end{cases}$$

The graph of this function is given on the next figure:



2 Transforms

Assume that X is a random variable. We can define a function $M_X(s)$ which is equal to the expectation of the random variable e^{sX}, where s is an argument of the function:

$$M_X(s) = \mathbf{E}\left[e^{sX}\right] \tag{3}$$

This function is called a **transform** or a **moment generating function** of a random variable X.

In discrete case, if X is a discrete random variable with PMF $p_X(x)$, we have:

$$M_X(s) = \mathbf{E}\left[e^{sX}\right] = \sum_x e^{sx} p_X(x). \tag{4}$$

Example 2.1. Let X be a discrete random variable with the following PMF:

$$p_X(x) = \begin{cases} 1/2, & x = 2 \\ 1/6, & x = 3 \\ 1/3, & x = 5 \\ 0, & \text{otherwise} \end{cases}$$

In this case, we have:

$$\begin{aligned} M_X(s) &= \mathbf{E}\left[e^{sX}\right] \\ &= \sum_x e^{sx} p_X(x) \\ &= e^{2s} \cdot \frac{1}{2} + e^{3s} \cdot \frac{1}{6} + e^{5s} \cdot \frac{1}{3}. \end{aligned}$$