Lecture 16

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1 Independence

Definition 1.1. *Two random variables X and Y are called **independent** if*

$$f_{X,Y}(x,y) = f_X(x) f_Y(y). \tag{1}$$

Since $f_{X|Y}(x|y) = f_{X,Y}(x,y)/f_Y(y)$, we have $f_{X,Y}(x,y) = f_{X|Y}(x|y) f_Y(y)$, and therefore, substituting into the definition of independence, we obtain

$$f_{X|Y}(x|y) = f_X(x), \tag{2}$$

which means that the distribution of X does not depend on the value, which the variable Y takes. If two random variables X and Y are independent, we have:

$$P(X \in A, Y \in B) = P(X \in A) P(Y \in B). \tag{3}$$

The properties of independent random variables are similar to the properties of independent discrete random variables. If X and Y are independent, we have:

$$\mathbf{E}[XY] = \mathbf{E}[X]\,\mathbf{E}[Y] \tag{4}$$

$$\mathbf{E}[g(X)h(Y)] = \mathbf{E}[g(X)]\,\mathbf{E}[g(Y)] \tag{5}$$

$$\mathbf{var}(X+Y) = \mathbf{var}(X) + \mathbf{var}(Y). \tag{6}$$

2 Joint CDF

We can define a **joint CDF** of two random variables in the similar way to the CDF of individual random variable. As always, CDFs will be denoted by capital letters:

$$F_{X,Y}(x,y) = P(X \le x, Y \le y) \tag{7}$$

Expressing CDF through PDF, we have the following equality:

$$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(t,w)\,dw\,dt. \tag{8}$$

Differentiating this equality, we have an expression of PDF in terms of CDF:

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}}{\partial x \partial y}(x,y). \tag{9}$$

Example 2.1. For two-dimensional uniform distribution in the unit square $(X, Y) \in [0, 1] \times [0, 1]$ we have that the probability of an event is equal to its area. Therefore,

$$F_{X,Y}(x, y) = P(X \leq x, Y \leq y) = xy \quad \text{for } x \in [0, 1], y \in [0, 1].$$

Taking a second partial derivative, we get the PDF of uniform distribution which is a constant:

$$\begin{aligned} f_{X,Y}(x, y) &= \frac{\partial^2 F_{X,Y}}{\partial x \partial y}(x, y) \\ &= \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} xy\right) \\ &= \frac{\partial}{\partial y}(y) \\ &= 1, \quad \text{for } x \in [0, 1], y \in [0, 1]. \end{aligned}$$

3 Derived distributions

In this section we will consider the distributions of the functions of random variables. The general problem will be stated as follows:

Assume X is a random variable with the PDF $f_X(x)$ (or CDF $F_X(x)$). Let Y be the random variable such that $Y = g(X)$. What is the distribution (PDF or CDF) of Y?

To answer this question we will first try to find the CDF of Y and then differentiate it to obtain PDF of Y. The CDF of Y is given by the following general formula:

$$F_Y(y) = P(Y \leq y) = P(g(X) \leq y) = \int_{\{x | g(x) \leq y\}} f_X(x)\, dx. \tag{10}$$

From this equality, the PDF of Y can be obtained:

$$f_Y(y) = \frac{dF_Y}{dy}(y). \tag{11}$$

Now let us consider several examples.

Example 3.1. Let X be a uniform random variable on the interval $[0, 1]$. Let $Y = \sqrt{X}$. We know the PDF and CDF of the uniform distribution. They are equal to:

$$f_X(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases} \qquad F_X(x) = \begin{cases} 0, & x \leq 0 \\ x, & 0 \leq x \leq 1 \\ 1, & 1 \leq x \end{cases}$$

We will first find the CDF of Y and then its PDF:

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(\sqrt{X} \leq y) \\ &= P(X \leq y^2) \\ &= F_X(y^2) = y^2, \quad \text{for } 0 \leq y \leq 1. \end{aligned}$$

Now, differentiating, we have:

$$f_Y(y) = 2y, \quad \text{for } 0 \leq y \leq 1.$$

Example 3.2. Now assume that X has some arbitrary PDF $f_X(x)$ and CDF $F_X(x)$, and $Y = X^2$. Again, first we will find the CDF of Y:

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(X^2 \leq y) \\ &= P(-\sqrt{y} \leq X \leq \sqrt{y}) \\ &= F_X(\sqrt{y}) - F_X(-\sqrt{y}). \end{aligned}$$

Differentiating the equality and using the chain rule, we get:

$$f_Y(y) = \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y}) = \frac{1}{2\sqrt{y}} \left(f_X(\sqrt{y}) + f_X(-\sqrt{y}) \right), \quad y \geq 0.$$

Now let's consider the special case when Y is a linear function of X, i.e.

$$Y = aX + b. \tag{12}$$

Consider the case when a is positive. We have:

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(aX + b \leq y) \\ &= P\left(X \leq \frac{y-b}{a}\right) \\ &= F_X\left(\frac{y-b}{a}\right). \end{aligned}$$

Differentiating and using the chain rule, we have:

$$f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right).$$

The case of negative a is similar. The general formula we have is the following:

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right). \tag{13}$$

Example 3.3. Let X be an exponentially distributed random variable with parameter λ:

$$f_X(x) = \lambda e^{-\lambda x}, \quad x \geq 0.$$

If $Y = aX + b$, we have:

$$f_Y(y) = \frac{\lambda}{|a|} e^{-\lambda(y-b)/a}, \quad \frac{y-b}{a} \geq 0.$$

Now we can see that in the special case when $a \geq 0$ and $b = 0$, we have:

$$f_Y(y) = \frac{\lambda}{a} e^{-\frac{\lambda}{a} y}, \quad y \geq 0,$$

and therefore the random variable $Y = aX$ is distributed exponentially with parameter $\frac{\lambda}{a}$.

Now let's consider the general case, when $Y = g(X)$, and let's assume that $g(x)$ is a strictly monotonic function. If the function is monotonic, then there exists a inverse function to it, i.e. we can express X in terms of Y. For example, if $Y = e^X$, then $X = \ln Y$. We will denote the inverse function $x = h(y)$. In this case we have the following fact.

Proposition 3.4. *If X is a random variable with the PDF $f_X(x)$, and*

$$Y = g(X), \qquad \textit{and} \quad X = h(Y), \tag{14}$$

then

$$f_Y(y) = f_X(h(y)) \left| \frac{dh}{dy}(y) \right|. \tag{15}$$

Example 3.5. Assume X is a uniform random variable on $[0, 1]$, and $Y = g(X) = e^X$. Therefore, $X = h(Y) = \ln Y$, and

$$\frac{dh}{dy}(y) = \frac{1}{y}.$$

The distribution of Y can be obtained using the proposition in the following way:

$$f_Y(y) = f_X(h(y)) \left| \frac{dh}{dy}(y) \right| = \frac{1}{y}, \quad 1 \leq y \leq e$$

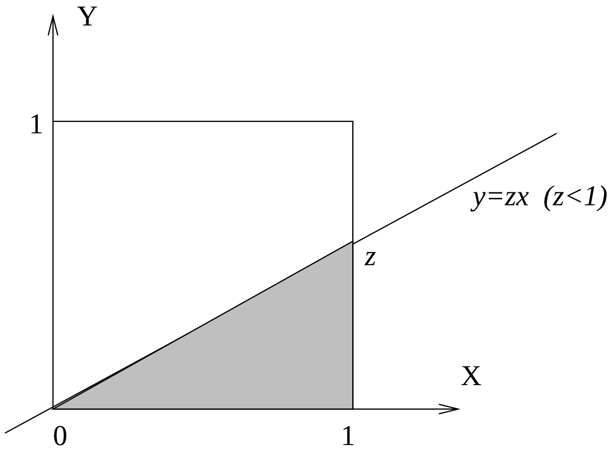
where we used that the PDF of X $f_X(x) = 1$ for $x \in [0, 1]$, and the range of Y is $[1, e]$, since range of X is $[0, 1]$, and $e^0 = 1$, and $e^1 = e$.

The same answer can be obtained without using the proposition in the similar manner as we did before.

Example 3.6. Let X and Y be uniform random variables on $[0, 1]$. Let $Z = Y/X$. We will find the distribution of Z. We have:

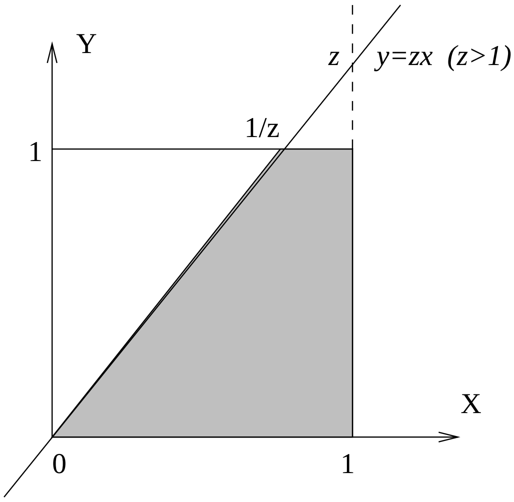
$$F_Z(z) = P\left(\frac{Y}{X} \leq z\right).$$

In case $z \leq 1$, the probability is equal to the following shaded area:



This is a right triangle with sides 1 and z, and therefore in case $z \leq 1$, $F_Z(z) = \frac{z}{2}$.

In case $z \geq 1$, the probability is equal to the following shaded area:



This is a trapezoid, area of which can be found by subtracting the area of the triangle with sides 1 and $1/z$ from the area of the entire square. Therefore, if $z \geq 1$, $F_Z(z) = 1 - \frac{1}{2z}$.

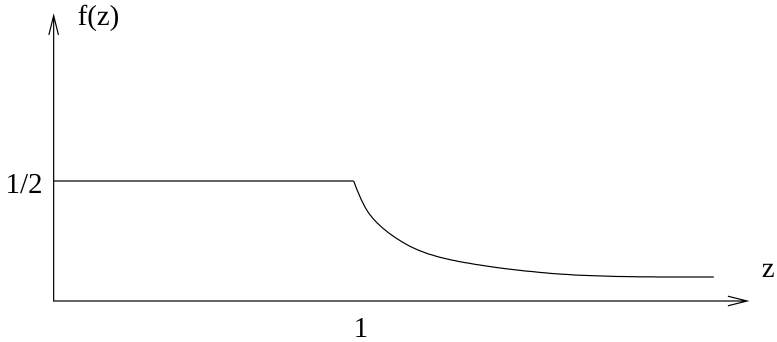
Combining these two results, we have the CDF of Z:

$$F_Z(z) = \begin{cases} z/2, & 0 \leq z \leq 1 \\ 1 - 1/(2z), & 1 \leq z. \end{cases}$$

Differentiating, we obtain the PDF of Z:

$$f_Z(z) = \begin{cases} 1/2, & 0 \leq z \leq 1 \\ 1/(2z^2), & 1 \leq z \\ 0, & \text{otherwise.} \end{cases}$$

This distribution has a shape, shown on the following figure:

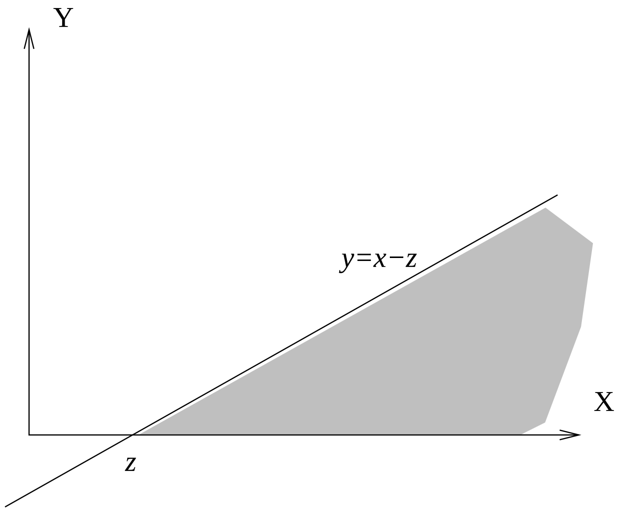


Example 3.7. Assume that X and Y are exponential random xariables with parameter λ. In

this case, $f_{X,Y}(x,y) = \lambda e^{-\lambda x} \times \lambda e^{-\lambda y}$. Let $Z = X - Y$. Let $z \geq 0$. Then we have:

$$\begin{aligned}
F_Z(z) &= P(Z \leq z) \\
&= P(X - Y \leq z) \\
&= 1 - P(X - Y \geq z) \\
&= 1 - \int_0^\infty \left[\int_{z+y}^\infty f_{X,Y}(x,y)\, dx \right] dy \\
&= 1 - \int_0^\infty \lambda e^{-\lambda y} \left[\int_{z+y}^\infty \lambda e^{-\lambda x}\, dx \right] dy \\
&= 1 - \int_0^\infty \lambda e^{-\lambda y} e^{-\lambda(z+y)}\, dy \\
&= 1 - e^{-\lambda z} \int_0^\infty \lambda e^{2\lambda y}\, dy \\
&= 1 - \frac{1}{2} e^{-\lambda z}.
\end{aligned}$$

The area of integration is drawn on the following figure:



Now we have to compute $F_Z(z)$ for $z \leq 0$. From symmetry, we have

$$F_Z(z) = P(Z \leq z) = P(-Z \geq -z) = P(Z \geq -z) = 1 - F_Z(-z).$$

Therefore, for $z \leq 0$, we have $F_Z(z) = \frac{1}{2} e^{\lambda z}$. Combining results, we have:

$$F_Z(z) = \begin{cases} 1 - \frac{1}{2} e^{-\lambda z}, & z \geq 0 \\ \frac{1}{2} e^{\lambda z}, & z \leq 0. \end{cases}$$

Now, differentiating, we obtain a PDF of Z:

$$f_Z(z) = \begin{cases} \frac{\lambda}{2} e^{-\lambda z}, & z \geq 0 \\ \frac{\lambda}{2} e^{\lambda z}, & z \leq 0. \end{cases}$$

This distribution is called a **Laplace distribution.**