Lecture 14

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1 Conditional expectation: conditioning on events

Now as we know what is a conditional PDF of the random variable (conditioned on event A) is, we can define the conditional expectation.

Definition 1.1. *A* ***conditional expectation*** *of the random variable X conditioned on the event A such that $P(A) > 0$ is*

$$\mathbf{E}[X|A] = \int_{-\infty}^{\infty} x f_{X|A}(x|A)\,dx. \tag{1}$$

For the conditional expectation the following form of the **total expectation theorem** holds. If events A_i form a partition of the universe, and the conditional expectations are known for all A_i we can compute the unconditional expectation of X:

$$\mathbf{E}[X] = \sum_i P(A_i)\mathbf{E}[X|A_i]. \tag{2}$$

Also, for continuous random variables, and partition of the universe into events, we have the following version of **total probability theorem:**

$$f_X(x) = \sum_i P(A_i) f_{X|A_i}(x|A_i). \tag{3}$$

In the next examples we will see how one can use them.

Example 1.2. Consider the random variable X, PDF of which is a piecewise constant, for example,

$$f_X(x) = \begin{cases} 1/3, & 0 \le x \le 1 \\ 2/3, & 1 < x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

There are two ways we can compute the expectation of X. First one is by direct computation:

$$\begin{aligned}\mathbf{E}[X] &= \int_{-\infty}^{\infty} f_X(x)\,dx \\ &= \int_0^2 f_X(x)\,dx \\ &= \int_0^1 \frac{1}{3}x\,dx + \int_1^2 \frac{2}{3}x\,dx \\ &= \frac{1}{3}\frac{x^2}{2}\Big|_0^1 + \frac{2}{3}\frac{x^2}{2}\Big|_1^2 \\ &= \frac{1}{3}\left(\frac{1}{2}\right) + \frac{2}{3}\left(\frac{4}{2} - \frac{1}{2}\right) \\ &= \frac{7}{6}.\end{aligned}$$

Another way is to use total expectation theorem. Let event A_1 and A_2 be as follows:

$$A_1 = \{0 \le X \le 1\}, \quad A_2 = \{1 < X \le 2\}.$$

Given event A_1 we see that PDF of X is a constant (on $[0,1]$), and therefore X is uniform on $[0,1]$. Thus, the conditional expectation of X conditioned on $A_1 = \{0 \le X \le 1\}$ is $\mathbf{E}[X|A_1] = (0{+}1)/2 = 1/2$. By similar reasoning, $\mathbf{E}[X|A_1] = (1{+}2)/2 = 3/2$. The probabilities of events A_1 and A_2 are the following:

$$P(A_1) = P(0 \le X \le 1) = \frac{1}{3}; \quad P(A_2) = P(1 < X \le 2) = \frac{2}{3}.$$

Therefore, by total expectation theorem, we have:

$$\mathbf{E}[X] = \mathbf{E}[X|A_1]\,P(A_1) + \mathbf{E}[X|A_2]\,P(A_2) = \frac{1}{2}\cdot\frac{1}{3} + \frac{3}{2}\cdot\frac{2}{3} = \frac{7}{6}.$$

Example 1.3. Assume the train comes to station every 15 minutes: 7:00, 7:15, 7:30, etc. You come to the station at some time, uniformly distributed between 7:10 and 7:30, and board the first available train. What is the distribution of your waiting time?

We can decompose the time into two periods: $A_1 = [7:10, 7:15]$, and $A_2 = [7:15, 7:30]$. In case of event A_1 you board the 7:15 train, in case of event A_2 you board the 7:30 train.

The probabilities of these events are the following:

$$P(A_1) = \frac{5}{20} = \frac{1}{4}; \quad P(A_2) = \frac{15}{20} = \frac{3}{4}.$$

In case of event A_1, the waiting time is uniformly distributed for 0 to 5 minutes, and in case of event A_2 – from 0 to 15 minutes. Therefore, the conditional PDF are the following:

$$f_{X|A_1}(x|A_1) = \begin{cases} 1/5, & 0 \le X \le 5 \\ 0, & \text{otherwise} \end{cases} \qquad f_{X|A_2}(x|A_2) = \begin{cases} 1/15, & 0 \le X \le 15 \\ 0, & \text{otherwise} \end{cases}$$

Using the total probability theorem, we have:

$$f_X(x) = P(A_1) f_{X|A_1}(x|A_1) + P(A_2) f_{X|A_2}(x|A_2).$$

Therefore,

$$f_X(x) = \begin{cases} \frac{1}{4}\cdot\frac{1}{5} + \frac{3}{4}\cdot\frac{1}{15}, & 0 \le x \le 5 \\ \frac{1}{4}\cdot 0 + \frac{3}{4}\cdot\frac{1}{15}, & 5 < x \le 15 \end{cases} = \begin{cases} \frac{1}{10}, & 0 \le x \le 5 \\ \frac{1}{20}, & 5 < x \le 15 \end{cases}$$