Lecture 13

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1 Distributions of maximum and minimum

Consider the situation in which you are taking three tests. The score on each test is a discrete uniform random variable taking values from 1 to 10. Your final score in this series of tests is determined as a maximum of three individual scores. What is the distribution of it?

Let X_1, X_2, X_3 be the individual test scores. The PMFs of them are:

$$f_{X_i}(x) = \begin{cases} 1/10, & X_i = 1, 2, \ldots, 10 \\ 0, & \text{otherwise.} \end{cases}$$

In this case the CDF of X_i's is:

$$F_{X_i}(k) = P(X_i \leq k) = P(X_i = 1) + \cdots + P(X_i = k) = \frac{k}{10}, \quad k = 1, 2, \ldots, 10.$$

Let

$$X = \max\{X_1, X_2, X_3\}.$$

We have:

$$\begin{aligned} F_X(k) &= P(X \leq k) \\ &= P(X_1 \leq k, X_2 \leq k, X_3 \leq k) \\ &= P(X_1 \leq k)P(X_2 \leq k)P(X_3 \leq k) \qquad \text{(because of independence)} \\ &= \left(\frac{k}{10}\right)^3. \end{aligned}$$

Now, using the formula

$$p_X(k) = F_X(k) - F_X(k-1), \tag{1}$$

we have:

$$p_X(k) = \left(\frac{k}{10}\right)^3 - \left(\frac{k-1}{10}\right)^3$$

Now let's consider the situation of the minimum of several random variables. Assume,

$$Y = \min\{X_1, X_2, X_3\}.$$

We have:

$$\begin{aligned}
F_Y(k) &= P(Y \le k) \\
&= 1 - P(Y > k) \\
&= 1 - P(X_1 > k, X_2 > k, X_3 > k) \\
&= 1 - P(X_1 > k)P(X_2 > k)P(X_3 > k) && \text{(because of independence)} \\
&= 1 - (1 - P(X_1 \le k))(1 - P(X_2 \le k))(1 - P(X_3 \le k)) \\
&= 1 - \left(1 - \left(\frac{k}{10}\right)\right)^3.
\end{aligned}$$

Now, using the same formula

$$p_X(k) = F_X(k) - F_X(k-1),$$

we have:

$$p_X(k) = \left(1 - \frac{k-1}{10}\right)^3 - \left(1 - \frac{k}{10}\right)^3$$

2 Normal random variable

The **normal random variable** is a random variable with PDF

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \tag{2}$$

where μ and σ are parameters of the distributions. The mean and the variance of the normal random variable are

$$\mathbf{E}[X] = \mu \tag{3}$$
$$\mathbf{var}(X) = \sigma^2 \tag{4}$$

If X is a normal random variable with parameters μ and σ^2, we will normally write

$$X \sim N(\mu, \sigma^2)$$

If $X \sim N(\mu, \sigma^2)$, and $Y = aX + b$, then we have

$$Y = aX + b \sim N(a\mu + b, a^2\sigma^2). \tag{5}$$

Often people consider the **standard normal random variable**, which is a normal variable with mean $\mu = 0$ and variance $\sigma^2 = 1$. The PDF of a standard normal random variable is

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}. \tag{6}$$

The CDF of the standard normal random variable is denoted by $\Phi(x)$:

$$\Phi(x) = P(X \le x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\, dt. \tag{7}$$

The values of the function $\Phi(x)$ can be found in the tables. The table gives values of Φ only for positive arguments. In case we need to find $\Phi(-x)$, we will proceed as following:

$$\begin{aligned}
\Phi(-x) &= P(X \leq -x) \\
&= P(X \geq x) && \text{(because X is symmetric about zero)} \\
&= 1 - P(X \leq x) \\
&= 1 - \Phi(x).
\end{aligned}$$

Therefore, we get the following rule:

$$\Phi(-x) = 1 - \Phi(x). \tag{8}$$

The following fact will allow us to find the probabilities associated with any normal random variables given the table for standard normal random variable. Assume

$$X \sim N(\mu, \sigma^2).$$

In this case,

$$Y = \frac{X - \mu}{\sigma} \sim N(0, 1). \tag{9}$$

In the following examples we will denote the standard normal random variable by Z.

Example 2.1. Assume the yearly accumulation of snow in inches in some state is distributed as a normal random variable with the mean $\mu = 60$ and variance $\sigma^2 = 20^2$. What is the probability that in a given year the snow accumulation will be more than 80 inches?

We have $X \sim N(60, 20^2)$, and we need to find $P(X \geq 80$. We have:

$$\begin{aligned}
P(X \geq 80) &= P(\frac{X - 60}{20} \geq \frac{80 - 60}{20}) \\
&= P(Z \geq 1) \\
&= 1 - \Phi(1).
\end{aligned}$$

From the table, $\Phi(1) = 0.8413$, and therefore,

$$P(X \geq 80) = 1 - \Phi(1) = 0.1587.$$

3 Conditioning on event

Conditional PDF of X, conditioned on event A, such that $P(A) > 0$ is a function $f_{X|A}(x)$ such that

$$P(X \in B|A) = \int_B f_{X|A}(x)\, dx. \tag{10}$$

In case the given event is $\{X \in A\}$ (we know that X belongs to some set A), we have:

$$\begin{aligned}
P(X \in B|X \in A) &= \frac{P(X \in B \cap X \in A)}{P(X \in A)} && (11) \\
&= \frac{\int_{A \cap B} f_X(x)\, dx}{\int_A f_X(x)\, dx} && (12)
\end{aligned}$$

Example 3.1 (Memoryless property of exponential distribution). Assume that the buses arrive to the bus stop with interarrival time exponentially distributed with parameter λ. I.e., if T is a random variable corresponding to the time between successive buses,

$$f_T(t) = \lambda e^{-\lambda t},$$

and therefore

$$P(T > t) = e^{-\lambda t}.$$

Assume you come to the bus stop at time $t \leq T$. We will find the distribution of the waiting time X, i.e. time you have to wait on the bus stop till the next bus arrives. We have:

$$X = T - t.$$

Let's first find the conditional CDF of X, conditioned on the time you come to the bus stop. Let $A = \{T > t\}$ – event meaning that you arrived to the bus stop at time t, which is less than T – time of the next arrival of the bus.

$$\begin{aligned}
F_{X|A}(x|A) &= P(X \leq x|A) \\
&= 1 - P(X > x|T > t) \\
&= 1 - P(T - t > x|T > t) \\
&= 1 - P(T > x + t|T > t) \\
&= 1 - \frac{P(T > x + t, T > t)}{P(T > t)} \\
&= 1 - \frac{P(T > x + t)}{P(T > t)} \\
&= 1 - \frac{e^{-\lambda(t+x)}}{e^{-\lambda t}} \\
&= 1 - e^{-\lambda x}.
\end{aligned}$$

So we see, that waiting time X is distributed as an exponential random variable with parameter λ and does not depend on the time t you came to the bus stop. It means, that no matter when you come to the bus stop, the distribution of your waiting time is always the same.

Saying it in different way, assume that the time to complete the operation is exponential random variable with parameter λ. As long as the operation has not been completed, remaining time has the same exponential CDF, no matter when we started.