Lecture 12

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1 General random variables

In a lot of experiments it is useful to consider the random variables which take not a discrete set of values, but a continuous. For example, the speed of the wind is a real number, a speed of the vehicle is a real number. All the theory developed for discrete random variables can be changed to accommodate this general situation. All the concepts introduced for discrete random variables have their counterparts for continuous. In this part of the course we will take a closer look at it.

2 Probability density function

Continuous random variables take values in some particular interval. Unlike discrete random variables, for continuous random variables we will talk about events in which the variable take a value in some given set. I.e. instead of events $\{X = x\}$ we will be considering events $\{X \in B\}$, where B is a subset of the real line. For example, if a random variable takes values from 0 to 1 we may talk about the probability that the random variables is greater than 0.5, which is $P(X > 0.5) = P(X \in (0.5, 1])$.

X is called a **continuous random variable** if its probability distribution can be described by a function $f_X(x)$ called a **probability density function (pdf)**, such that

$$P(X \in B) = \int_B f_X(x)\,dx. \tag{1}$$

In case when $B = [a, b]$, we have:

$$P(X \in B) = P(a \le X \le b) = \int_a^b f_X(x)\,dx. \tag{2}$$

Let us notice, that

$$P(X = a) = \int_a^a f_X(x)\,dx = 0, \tag{3}$$

and therefore,

$$P(a \le X \le b) = P(a < X < b) = P(a \le X < b) = P(a < X \le b). \tag{4}$$

Geometrically speaking, the probability that X belong to the interval $[a, b]$ is the area under the graph of the function $f_X(x)$ above the interval $[a, b]$.

Furthermore, we can see that $P(-\infty < X < \infty) = 1$, and therefore pdf should have a following property:

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1. \tag{5}$$

Now let's look at the physical interpretation of $f_X(x)$. Let δ be a small number. Therefore,

$$P([x, x+\delta]) = \int_{x}^{x+\delta} f_X(t)\,dt \approx \delta f_X(x). \tag{6}$$

So, the $f_X(x)$ is a probability mass per unit length.

Now we will consider several examples.

Example 2.1 (Continuous Uniform Random Variable). Let us consider a random variable which takes values on the interval $[0, 1]$ such that all subsets of the same length are equally likely. Therefore, the pdf of this random variable is constant:

$$f_X(x) = \begin{cases} c, & x \in [0,1] \\ 0, & \text{otherwise} \end{cases}$$

The constant c can be determined from the normalization property:

$$1 = \int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^1 c\,dx = c.$$

Therefore, the pdf of the discrete random variable taking values on the interval $[0, 1]$ is given by:

$$f_X(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & \text{otherwise} \end{cases} \tag{7}$$

In a general case, when the uniform random variable takes values on the interval $[a, b]$, we have

$$f_X(x) = \begin{cases} \dfrac{1}{b-a}, & x \in [a,b] \\ 0, & \text{otherwise} \end{cases} \tag{8}$$

Example 2.2. The pdf of the random variable is not necessarily bounded: let's consider the random variable with the following pdf

$$f_X(x) = \begin{cases} \dfrac{1}{2\sqrt{x}}, & x \in [0,1] \\ 0, & \text{otherwise} \end{cases}$$

This is a valid pdf, since we have:

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^1 \frac{1}{2\sqrt{x}}\,dx = \sqrt{x}\Big|_0^1 = 1,$$

but it can take arbitrarily large values.

3 Expectation

The **expectation** for continuous random variables can be defined in the similar way as for discrete random variables:

$$\mathbf{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx. \tag{9}$$

It is an anticipated average value in large number of independent repetitions of experiment. In case $Y = g(X)$, we have

$$\mathbf{E}[Y] = \mathbf{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx. \tag{10}$$

We can define n**-th moment** as

$$\mathbf{E}[X^n] = \int_{-\infty}^{\infty} x^n f_X(x)\, dx. \tag{11}$$

The **variance** is

$$\mathbf{var}(X) = \mathbf{E}\left[(X - \mathbf{E}[X])^2\right] = \int_{-\infty}^{\infty} (x - \mathbf{E}[X])^2 f_X(x)\, dx. \tag{12}$$

The following formula for variance is also true for continuous random variables:

$$\mathbf{var}(X) = \mathbf{E}\left[X^2\right] - (\mathbf{E}[X])^2. \tag{13}$$

Example 3.1. Let's compute the expectation and variance of the continuous uniform random variable:

$$\begin{aligned}\mathbf{E}[X] &= \int_{-\infty}^{\infty} x f_X(x)\, dx \\ &= \int_a^b x \frac{1}{b-a}\, dx \\ &= \frac{1}{b-a} \left.\frac{x^2}{2}\right|_a^b \\ &= \frac{b^2 - a^2}{2(b-a)} = \frac{a+b}{2}.\end{aligned}$$

$$\begin{aligned}\mathbf{E}\left[X^2\right] &= \int_a^b \frac{x^2}{b-a}\, dx \\ &= \frac{a^2 + ab + b^2}{3}.\end{aligned}$$

$$\mathbf{var}(X) = \frac{(b-a)^2}{12}$$

Example 3.2. Let X be a uniform random variable on $[0, 1]$. Assume

$$Y = g(X) = \begin{cases} 1, & x \leq 1/3 \\ 2, & x > 1/3 \end{cases}$$

In this case Y is a discrete random variable with the following PMF:

$$p_Y(y) = \begin{cases} 1/3, & y = 1 \\ 2/3, & y = 2 \end{cases}$$

Therefore,

$$\mathbf{E}[Y] = 1 \cdot \frac{1}{3} + 2 \cdot \frac{2}{3} = \frac{5}{3}.$$

Also, we can compute it in a different way:

$$\begin{aligned} \mathbf{E}[Y] = \mathbf{E}[g(X)] &= \int_0^1 g(x) f_X(x)\, dx \\ &= \int_0^{1/3} dx + \int_{1/3}^1 2\, dx = \frac{5}{3}. \end{aligned}$$

4 Exponential random variable

Exponential random variable is a random variable with the following pdf:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases} \tag{14}$$

We can compute the expectation of the exponential random variable and its variance.

$$\begin{aligned} \mathbf{E}[X] &= \int_0^\infty x \lambda e^{-\lambda x}\, dx \\ &= \left(-x e^{-\lambda x}\right)\Big|_0^\infty + \int_0^\infty e^{-\lambda x}\, dx \\ &= 0 - \left.\frac{e^{-\lambda x}}{\lambda}\right|_0^\infty \\ &= \frac{1}{\lambda} \end{aligned}$$

$$\begin{aligned} \mathbf{E}\left[X^2\right] &= \int_0^\infty x^2 \lambda e^{-\lambda x}\, dx \\ &= \left(-x^2 e^{-\lambda x}\right)\Big|_0^\infty + \int_0^\infty 2x e^{-\lambda x}\, dx \\ &= 0 + \frac{2}{\lambda}\mathbf{E}[X] \\ &= \frac{2}{\lambda^2}. \end{aligned}$$

Finally, using the formula for $\mathbf{var}(X)$, we get:

$$\mathbf{var}(X) = \mathbf{E}\left[X^2\right] - (\mathbf{E}[X])^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$$

The exponential random variable is normally used for time, until equipment breaks down, time until the accident, etc. It will play a crucial role by the end of the course when we will be talking about the Poisson processes.

5 Cumulative density function

So far, discrete and continuous random variables are described by different objects: PMF in case of discrete random variables, and pdf in case of continuous random variables. It is good to develop a way to describe all random variables by means independent of whether random variable is discrete or continuous.

Definition 5.1. ***Cumulative density function (CDF)*** *of the random variable X is defined as*

$$F_X(x) = P(X \leq x) = \begin{cases} \sum\limits_{k \leq x} p_X(k), & X: \textit{ discrete} \\ \int_{-\infty}^{x} f_X(t)\, dt, & X: \textit{ continuous} \end{cases} \tag{15}$$

We will list the properties of the CDF:

(1) $F_X(x)$ is nondecreasing:

$$F_X(x) \leq F_X(y) \text{ for } x \leq y;$$

(2) Limit properties:

$$F_X(x) \to 0 \text{ as } x \to -\infty \tag{16}$$

$$F_X(x) \to 1 \text{ as } x \to \infty \tag{17}$$

(3) For discrete random variable X,

$$p_X(k) = P(X \leq k) - P(X \leq k-1) = F_X(k) - F_X(k-1). \tag{18}$$

(4) For continuous random variable X,

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt, \tag{19}$$

and

$$f_X(x) = \frac{dF_X(x)}{dx}. \tag{20}$$

In the next example, we will find the CDF of geometric random variable and of exponential random variable, and look at the connection between them.

Example 5.2 (CDF of geometric and exponential random variables). For geometric random variable we have:

$$p_X(k) = p(1-p)^{k-1},$$

therefore,

$$F_X^{\text{geo}}(n) = P(X \leq n) = \sum_{k=1}^{n} p(1-p)^{k-1} = p\frac{1-(1-p)^n}{1-(1-p)} = 1-(1-p)^n$$

Now, for exponential random variable, we have:

$$F_X^{\text{exp}}(x) = \int_0^x \lambda e^{-\lambda t}\, dt = 1 - e^{-\lambda x}. \tag{21}$$

Let $\delta = \dfrac{-\ln(1-p)}{\lambda}$. Therefore,

$$(1-p) = e^{-\lambda\delta}.$$

If δ is small, we get the proximity between exponential distribution and geometric. Exponential distribution can be viewed as a limiting case of the following situation: every δ seconds a coin with probability of heads equal to p – small, is flipped. The number of the experiment on which we get the first head is given by the geometric distribution. The time of the occurrence of the first head can be approximated by the exponential distribution.