Lecture 4

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1 Independence

For the given event B let's consider events A such that the occurrence of B doesn't alter the probability of A, i.e.

$$P(A|B) = P(A).$$

In this case we will be talking about A being independent on B. Rewriting the conditional probability, we obtain

$$\frac{P(A \cap B)}{P(B)} = P(A),$$

from where

$$P(A \cap B) = P(A)P(B).$$

This relation is symmetric, and thus, if A is independent on B, we see that B is also independent on A. That allows us to talk about the independent events.

Definition 1.1. *Events A and B such that*

$$P(A \cap B) = P(A)P(B) \tag{1}$$

*are called **independent**.*

Sometimes it is easy to understand intuitively: if two events are governed by two unrelated processes, the events are independent. It is difficult, though, to understand it graphically. For example, two events which are disjoint are not independent, since if A and B are disjoint such that $P(A) > 0$ and $P(B) > 0$, then $P(A \cap B) = 0$, but $P(A)P(B) > 0$.

Example 1.2. *Consider two rolls of a fair 4-sided die. Let the event A be the event of getting number 2 on the first roll, and the event B be the event of getting number 3 on the second roll. Then, $A = \{(2,1),(2,2),(2,3),(2,4)\}$ and $B = \{(1,3),(2,3),(3,3),(4,3)\}$. Therefore, $P(A) = 4/16 = 1/4$, $P(B) = 4/16 = 1/4$, and since $A \cap B = \{(2,3)\}$ and $P(A \cap B) = 1/16 = P(A)P(B)$, events A and B are independent.*

Example 1.3. *Now let event A be the event of getting 1 on the first roll, and the event B be the event of getting the sum of two rolls equal to 5. Then, $A = \{(1,1),(1,2),(1,3),(1,4)\}$ and $B = \{(1,4),(2,3),(3,2),(4,1)\}$. Therefore, $P(A) = P(B) = 1/4$. Furthermore, $A \cap B = \{(1,4)\}$, and $P(A \cap B) = 1/16 = P(A)P(B)$. Thus, A and B are independent.*

Example 1.4. *Now let the event A be the event, that the maximum of 2 rolls is 2, and the event B be the event, that the minimum of 2 rolls is 2. In this case,* $A = \{(1,2),(2,2),(2,1)\}$, $B = \{(2,2),(2,3),(2,4),(3,2),(4,2)\}$, $P(A) = 3/16$, *and* $P(B) = 5/16$. *Furthermore,* $A \cap B = \{(2,2)\}$, *and thus* $P(A \cap B) = 1/16 \neq P(A)P(B)$. *Thus, events A and B are not independent.*

The definition of the independence of 2 events can be generalized to the case of 3 or more events:

Definition 1.5. *A collection of events $A_1, A_2, \ldots, A_n$ is called* ***independent*** *if for any subset of indices* $S \subset \{1, 2, \ldots, n\}$,

$$P(\bigcap_{i \in S} A_i) = \prod_{i \in S} P(A_i). \tag{2}$$

In case of three events, the definition implies the following four equalities:

$$\begin{aligned} P(A_1 \cap A_2) &= P(A_1)P(A_2); \\ P(A_1 \cap A_3) &= P(A_1)P(A_3); \\ P(A_2 \cap A_3) &= P(A_2)P(A_3); \\ P(A_1 \cap A_2 \cap A_3) &= P(A_1)P(A_2)P(A_3). \end{aligned}$$

In the next two examples we will see that the first three equalities do no follow from the last one, and the last one does not follow from the first three.

Example 1.6 (Pairwise independence is not enough for independence). *Let's consider an experiment of tossing a fair coin twice. Let event H_1 be the event of obtaining HEAD on the first toss, the event H_2 be the event of obtaining HEAD on the second toss, and the event D be the event of getting different results on two tosses. Therefore,* $H_1 = \{HH, HT\}, H_2 = \{HH, TH\}, D = \{HT, TH\}$. *Obviously, H_1 and H_2 are independent. Moreover,*

$$P(D|H_1) = \frac{P(H_1 \cap D)}{P(H_1)} = \frac{1/4}{1/2} = \frac{1}{2} = P(D)$$

and thus, H_1 and D are independent. By the similar reasoning, H_2 and D are independent. But

$$P(H_1 \cap H_2 \cap D) = 0 \neq P(D)P(H_1)P(H_2) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}.$$

Example 1.7 (Equality $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$ is not enough for independence). *Let's consider two rolls of a fair 6-sided die. Let the events be the following:* $A = \{\textit{first roll is 1, 2, or 3}\}, B = \{\textit{first roll is 3, 4, or 5}\}$, *and* $C = \{\textit{sum of two rolls is 9}\} = \{(3,6),(6,3),(4,5),(5,4)\}$. *Then*

$$P(A) = \frac{1}{2}, \quad P(B) = \frac{1}{2}, \quad P(C) = \frac{1}{9}.$$

Furthermore,

$$\begin{aligned} P(A \cap B) &= \frac{1}{6} \neq P(A)P(B) = \frac{1}{2} \cdot \frac{1}{2}; \\ P(A \cap C) &= \frac{1}{36} \neq P(A)P(C) = \frac{1}{2} \cdot \frac{1}{9}; \\ P(B \cap C) &= \frac{3}{36} \neq P(B)P(C) = \frac{1}{2} \cdot \frac{1}{9}. \end{aligned}$$

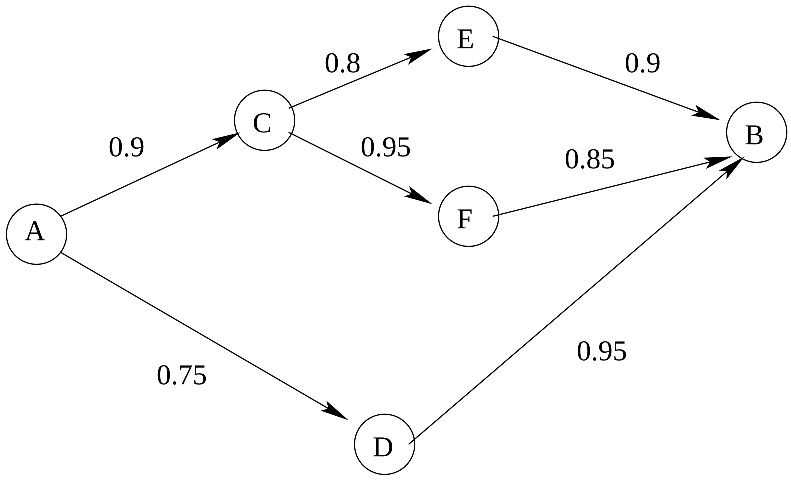
but

$$P(A \cap B \cap C) = \frac{1}{36} = P(A)P(B)P(C) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{9}.$$

These two examples serve to show that the first three equalities do not follow from the last one, and the last one do not follow from the first three.

2 Reliability issues

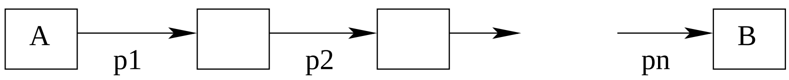
Let's consider the following network.



The information should be transferred from node A to node B. The links of the network are functional with the probabilities specified. The information from A to B is transferred correctly in case there is at least one fully working path from A to B. What is the probability that the information will be transferred?

Before we get to this problem, let's consider two special cases.

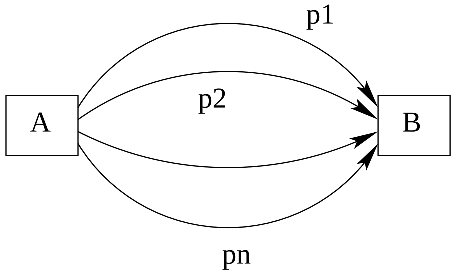
Sequential connection Assume the nodes are connected sequentially:



In this case the probability that the information will be transferred from node A to node B is equal

$$p_{AB} = p_1 p_n \dots p_n. \tag{3}$$

Parallel connection Assume there are several parallel connections:



In this case, we can also compute the probability that the information will be transferred from A to B:

$$\begin{aligned} P(\{\text{the information is transferred}\}) &= 1 - P(\{\text{the information is not transferred}\}) = \\ &= 1 - P(\{\text{all links are broken}\}) = \\ &= 1 - P(\{\text{link 1 is broken}\}) \dots P(\{\text{link } n \text{ is broken}\}) = \\ &= 1 - (1 - p_1) \dots (1 - p_n). \end{aligned}$$

Therefore, in this case

$$p_{AB} = 1 - (1 - p_1)(1 - p_2) \dots (1 - p_n). \tag{4}$$

These formulae above can be used to solve the original problem. In order to do that, we need to decompose the initial network into several elementary parallel and sequential connections.

$$\begin{aligned} p_{CEB} &= p_{CE} p_{EB} = 0.8 \cdot 0.9 = 0.72; \\ p_{CFB} &= p_{CF} p_{FB} = 0.95 \cdot 0.85 = 0.8075; \\ p_{CB} &= 1 - (1 - p_{CEB})(1 - p_{CFB}) = 0.9461; \\ p_{ACB} &= p_{AC} p_{CB} = 0.9 \cdot 0.9461 = 0.851; \\ p_{ADB} &= p_{AD} p_{DB} = 0.75 \cdot 0.95 = 0.7125; \\ p_{AB} &= 1 - (1 - p_{ACB})(1 - p_{ADB}) = 1 - (1 - 0.851)(1 - 0.7125) = 0.957. \end{aligned}$$

Thus, the probability, that the information will be transferred from A to B is $p_{AB} = 0.957$.