Lecture 3

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1 Multiplication rule

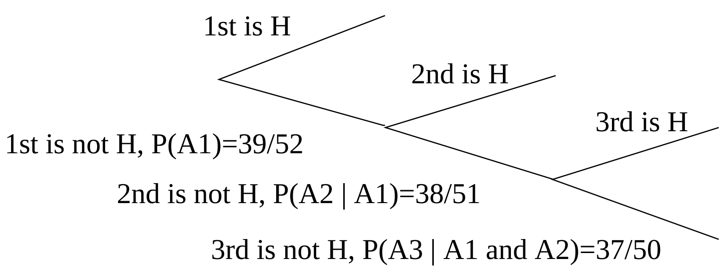
First we will start with an example:

Example 1.1. *Assume, that we draw at random 3 card out of the standard deck of 52 cards. What is the probability, that none of these cards will be a heart?*

The probability that the first card is not a heart is 39/52 (there is total of 52 cards, 39 oh which are not hearts). Given that the first card was not a heart, the second card is not a heart with probability 38/51 (38 not hearts are left in the deck of 51 cards). Given that the first and the second card are not hearts, the probability for the third card not to be a heart is 37/50 (37 not hearts left in the deck of 50 cards). Thus, the probability of 3 not hearts being drawn from the deck is equal to

$$P(\textit{3 not hearts}) = \frac{39}{52} \cdot \frac{38}{51} \cdot \frac{37}{50}.$$

This situation can be represented by the following tree structure.



Two leftmost branches correspond to the first card, which can be a not-heart with a probability of $P(A_1) = 39/52$ *(downward branch) and can be a heart with a probability of* $P(A_1^c) = 13/52$ *(upward branch). After the event* A_1 *of not getting a heart happened, there are two other possibilities. The second card can be a not-heart (second downward branch) with probability* $P(A_2|A_1) = 38/51$*, and it can be a heart (second upward branch). Finally, the third card has also three choices, after two non-hearts were drawn. It can be a non-heart (third downward branch) with a probability* $P(A_3|A_1 \cap A_2) = 37/50$ *or it can be a heart (third upward branch).*

In this problem we had to find $P(A_1 \cap A_2 \cap A_3)$*. Thus, substituting all the probabilities, we see that*

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2)$$

This formula can be generalized in the following way:

Theorem 1.2 (Multiplication rule). *Assume* $A_1, A_2, \dots, A_n$ *are events such that* $P(A_i) \neq 0$*. Then*

$$P(A_1 \cap A_2 \cap \dots \cap A_n) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \dots P(A_n|A_1 \cap \dots \cap A_{n-1}). \quad (1)$$

Proof. Let's use the definition of the conditional probability for the right hand of the above equality:

$$P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \dots P(A_n|A_1 \cap \dots \cap A_{n-1}) =$$
$$P(A_1)\frac{P(A_1 \cap A_2)}{P(A_1)}\frac{P(A_1 \cap A_2 \cap A_3)}{P(A_1 \cap A_2)} \dots \frac{P(A_1 \cap \dots \cap A_{n-1} \cap A_n)}{P(A_1 \cap \dots \cap A_{n-1})} =$$
$$P(A_1 \cap \dots \cap A_{n-1} \cap A_n)$$

since everything, except the last numerator cancels out. □

We will use multiplication rule in the following example.

Example 1.3. *Assume there are 16 people in the class, 12 girls and 4 boys. The students are divided into 4 groups of 4 people. What is the probability that each group will have exactly one boy?*

Let's consider the following events:

$$A_1 = \{\textit{1st and 2nd boys are in different groups}\}$$
$$A_2 = \{\textit{1st, 2nd and 3rd boys are in different groups}\}$$
$$A_3 = \{\textit{1st, 2nd, 3rd and 4th boys are in different groups}\}$$

We need to compute the probability of A_3*, for which the following equality holds (second equality is from multiplication rule)*

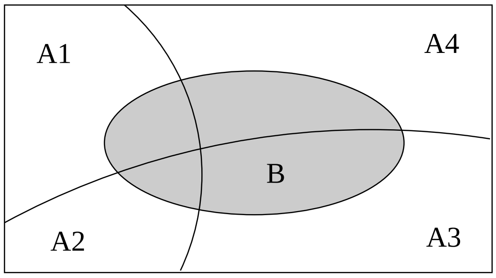
$$P(A_3) = P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2).$$

Let's compute all these probabilities. A_1 *is the event that boys 1 and 2 are in different groups. Let's fix the position of the first boy. After that, there are total of 15 open slots for a second boy, out of which 12 are not in the same group as the first boy. Thus,* $P(A_1) = 12/15$*. Now, assume that boys 1 and 2 are already in different groups. For the 3rd boy there are total of 14 open slots, and only 8 of them are not in the groups of the 1st or the 2nd boys. Thus,* $P(A_2|A_1) = 8/14$*. Now, the 4th boy has only 13 open slots, and only 4 of them are not in the groups of 1st, 2nd and 3rd boys. Thus,* $P(A_3|A_1 \cap A_2) = 4/13$*. Thus, the answer to the problem is*

$$P(A_1 \cap A_2 \cap A_3) = \frac{12}{15} \cdot \frac{8}{14} \cdot \frac{4}{13}.$$

2 Total probability theorem

Assume the situation, illustrated on the following picture.



Let B be an event, and events $A_1, \dots, A_n$ form a partition of the universe Ω (i.e. they are disjoint, and their union is equal to Ω). Then, event B can be represented as a union of its parts, lying in different A_i's:

$$B = (B \cap A_1) \cup (B \cap A_2) \cup \dots \cup (B \cap A_n).$$

The events in the right hand side of this equality are disjoint, and thus

$$P(B) = P(B \cap A_1) + P(B \cap A_2) + \dots + P(B \cap A_n).$$

The probabilities in the right hand side can be written using the conditional probability definition: $P(B \cap A_i) = P(A_i)P(B|A_i)$, which will lead to the following equality, known as **total probability theorem:**

$$P(B) = P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + \dots + P(A_n)P(B|A_n). \tag{2}$$

We will demonstrate the use of it in the following example.

Example 2.1. *Assume you are participating in the chess tournament. You know that you have a probability of winning 0.3 against 1/2 of the players, 0.4 against 1/4 of the players, and 0.5 against the remaining 1/4 of the players. You are playing one game with a random person. What is the probability of winning?*

Let the events be the following: A_1, A_2, A_3 *– events of playing with a person from 1st, 2nd, or 3rd group correspondingly, and* W *be the event of winning the game. We are given:*

$$\begin{aligned} P(A_1) &= 1/2; & P(W|A_1) &= 0.3 \\ P(A_2) &= 1/4; & P(W|A_2) &= 0.4 \\ P(A_3) &= 1/4; & P(W|A_3) &= 0.5 \end{aligned}$$

By total probability theorem,

$$P(W) = P(A_1)P(W|A_1) + P(A_2)P(W|A_2) + P(A_3)P(W|A_3) = \frac{1}{2} \cdot 0.3 + \frac{1}{4} \cdot 0.4 + \frac{1}{4} \cdot 0.5 = 0.375.$$

The following example is a brief introduction to what is called **Markov chains**. We will talk about them in greater details by the end of the course.

Example 2.2. *Assume that the probability that if the stock price went up today, the stock price will go up tomorrow is 0.8, and it will go down tomorrow is 0.2. If the stock price went down today, it will go up tomorrow with probability 0.6 and go down tomorrow with probability 0.4.*

Assume that the stock price is going up today. What is the probability that the stock price will go up in 3 days?

The events we will use are the following: U_i *– event of the stock price going up after* i *days, and* D_i *– event of the stock price going down in* i *days. Thus, we need to find the probability* $P(U_3)$.

By law of total probability, conditioning on the behavior of the stock on the second day, we get:

$$\begin{aligned} P(U_3) &= P(U_2)P(U_3|U_2) + P(D_2)P(U_3|D_2) \\ &= P(U_2) \cdot 0.8 + P(D_2) \cdot 0.6. \end{aligned}$$

Now, similarly,

$$\begin{aligned} P(U_2) &= P(U_1)P(U_2|U_1) + P(D_1)P(U_2|D_1) \\ &= P(U_1) \cdot 0.8 + P(D_1) \cdot 0.6; \\ P(D_2) &= P(U_1)P(D_2|U_1) + P(D_1)P(D_2|D_1) \\ &= P(U_1) \cdot 0.2 + P(D_1) \cdot 0.4; \end{aligned}$$

Since today the stock price went up, we have

$$\begin{aligned} P(U_1) &= 0.8; \\ P(D_1) &= 0.2. \end{aligned}$$

Substituting, we can obtain the answer to the problem.

3 Bayes' rule

Assume $A_1, A_2, \ldots, A_n$ form a partition of the universe, and $P(A_i) \neq 0$. Assume B is an event. We have:

$$P(B|A_i) = \frac{P(B \cap A_i)}{P(A_i)},$$

and

$$P(A_i|B) = \frac{P(B \cap A_i)}{P(B)}.$$

Expressing $P(B \cap A_i)$ from these two equations, we get the following equality:

$$P(B)P(A_i|B) = P(A_i)P(B|A_i),$$

from where

$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{P(B)}.$$

Substituting the expression for $P(B)$ obtained from the law of total probability, we obtain **Bayes' rule:**

$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + \cdots + P(A_n)P(B|A_n)}. \quad (3)$$

The Bayes' rule is used for inference. Assume, we know death rates from heart attack and from the brain tumor ($P(\text{death}|\text{heart attack})$ and $P(\text{death}|\text{brain tumor})$). Assume that we know the probability of getting either of these diseases ($P(\text{heart attack})$ and $P(\text{brain tumor})$). By law of total probability we can calculate the probability of the death. The Bayes' rule is used in other way. Assume we know that a person died. What is the probability that the cause of his death was a heart attack? a brain tumor?

Thus, we apply Bayes' rule if we want to know something about the cause, after we observed the effect. The events $A_1, \ldots, A_n$ correspond to different causes, and the event B corresponds to an observed effect. Let's demonstrate this in the following examples.

Example 3.1. *Let's get back to the radar problem from the previous lecture. The events were A – aircraft is present, and R – radar registered the aircraft. The given probabilities were:*

$$P(A) = 0.05;$$
$$P(R|A) = 0.99;$$
$$P(R|A^c) = 0.1.$$

Assume that the radar registered aircraft. What is the probability, that the aircraft is really there? By Bayes' rule

$$P(A|R) = \frac{P(A)P(R|A)}{P(A)P(R|A) + P(A^c)P(R|A^c)} = \frac{0.05 \cdot 0.99}{0.05 \cdot 0.99 + 0.95 \cdot 0.1} \approx 0.35.$$

Example 3.2. *Let's get back to the chess tournament. We had:*

$$P(A_1) = 1/2; \qquad P(W|A_1) = 0.3$$
$$P(A_2) = 1/4; \qquad P(W|A_2) = 0.4$$
$$P(A_3) = 1/4; \qquad P(W|A_3) = 0.5$$

Assume, you won the game. What is the probability, that you played with the participant from the first group? By Bayes' rule,

$$P(A_1|W) = \frac{P(A_1)P(W|A_1)}{P(A_1)P(W|A_1) + P(A_2)P(W|A_2) + P(A_3)P(W|A_3)} =$$
$$= \frac{1/2 \cdot 0.3}{1/2 \cdot 0.3 + 1/4 \cdot 0.4 + 1/4 \cdot 0.5} = 0.4$$