Lecture 2

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1 Examples of Probabilistic Models

1.1 Discrete Models

Example 1.1. *Consider an experiment, when we toss a fair coin once. There are two possible elementary outcomes: head (H) and tail (T). Thus, the sample space is* $\Omega = \{H, T\}$*. The possible events are:*

$$\{H, T\}, \{H\}, \{T\}, \emptyset.$$

Since the coin is fair, $P(\{H\}) = P(\{T\})$ *– the head and tail occur equally likely. Moreover,* $P(\{H, T\}) = P(\{H\}) + P(\{T\}) = 1$*, since that's the probability of the whole universe. From here,* $P(\{H\}) = P(\{T\}) = 0.5$*. Also,* $P(\emptyset) = 0$*. Such specified* P *gives us a probability law in case of a fair coin.*

Example 1.2. *Consider an experiment, consisting of three tosses of a fair coin. In this case the sample space has 8 equally likely outcomes:*

$$\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

In this case the probability of each outcome should be equal to $1/8$*. This allows us to find the probabilities of more complicated events, like event* A *of having exactly 2 heads. This event is a following set:* $\{HHT, HTH, THH\}$*. In order to calculate the probability of it, we will use an addition axiom:*

$$P(\{HHT, HTH, THH\}) = P(\{HHT\}) + P(\{HTH\}) + P(\{THH\}) = 3/8.$$

This example can be generalized. In case of event $A = \{s_1, s_2, \ldots, s_n\}$, it's probability can be computed by adding the probabilities of elementary outcomes, of which the event consists. This is called **discrete probability law:**

$$P(\{s_1, s_2, \ldots, s_n\}) = P(\{s_1\}) + P(\{s_2\}) + \cdots + P(\{s_n\}). \tag{1}$$

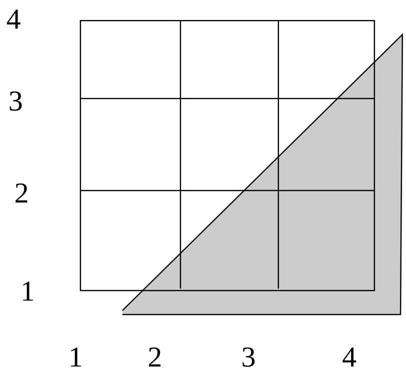
In case all the outcomes of the events are equally likely, and there is a total of n possible elementary outcomes, each of them has probability $1/n$, and for any event A,

$$P(A) = \frac{\text{number of elements in } A}{n}. \tag{2}$$

This formula is called **discrete uniform probability law**.

As an illustration of this formula we will consider one more example.

Example 1.3. *Assume a 4-sided die is rolled twice. Thus, the possible outcomes are pairs (i, j), where $i, j = 1, \ldots, 4$, and there is 16 possible outcomes. Since all of them are equally likely, for any pair (i, j), $P(\{(i, j)\}) = 1/16$. The outcomes can be represented as a point of 4×4-grid, where x-coordinate represents a first obtained number, and y-coordinate represents a second number.*



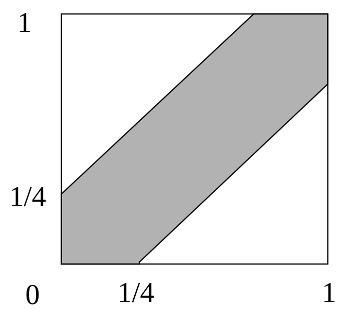
In order to compute the probability of the event $A = \{$1st number is greater than 2nd number$\}$, we have to count the points of the grid, for which it is true. In total there are 6 such point (all of them are within the shaded area on the picture), thus, the probability of this event is $P(A) = 6/16 = 3/8$.

1.2 Continuous Models

Example 1.4. *Assume a random number is drawn from the interval $\Omega = [0, 1]$. If the number is drawn uniformly, what would be the probability of getting some particular number? I.e. what is the probability of the event, consisting of a single element? If it is positive, then the events with sufficiently large number of elements would have a probability greater than 1, which is impossible. Thus, the probability of obtaining a given number is equal to 0.*

It makes sense to define a probability of a subset of $\Omega = [0, 1]$ as a "length" of it. For example, $P([a, b]) = b - a$. That will give a legitimate probability law.

Example 1.5. *Romeo and Juliet have a date at a given time. Each of them arrive with a delay between 0 and 1 hour, with all pairs of delays being equally likely. The first person to arrive waits for 15 minutes, and leaves if the other has not arrived. What would be the probability of a successful date?*



The sample space here will be the square $\Omega = [0, 1] \times [0, 1]$, the points of which denote the pair of delays of Romeo and Juliet. All pairs are equally likely means, that the probability of a subset of Ω is equal to its area. The event that Romeo meets Juliet is given by the set

$A = \{(x,y)|0 \le x \le 1, 0 \le y \le 1, |x-y| \le 1/4\}$ (shaded on the picture). The area of the shaded area is equal to the difference between 1 and sum of the area of 2 white triangles: $1 - 2 \cdot (3/4)^2/2 = 1 - 9/16 = 7/16$. Thus, the probability of a successful date is $P(A) = 7/16$.

2 Further Properties of Probability Law

The following properties of probability law can be derived from the probability axioms:

a). If $A \subset B$, then $P(A) \le P(B)$.

b). $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

c). The direct consequence of the previous property: $P(A \cup B) \le P(A) + P(B)$.

d). Generalization of the previous property: $P(A_1 \cup \cdots \cup A_n) \le P(A_1) + \cdots + P(A_n)$.

Proof. a). In case $A \subset B$, the event B can be decomposed into union of two disjoint events in the following way (that can be seen from Venn's diagrams):

$$B = A \cup (A^c \cap B).$$

Therefore, $P(B) = P(A) + P(A^c \cap B)$, and thus $P(A) \le P(B)$.

b). The event $A \cup B$ can be decomposed into union of two disjoint events in the following way:

$$A \cup B = A \cup (A^c \cap B),$$

and therefore

$$P(A \cup B) = P(A) + P(A^c \cap B).$$

The event B can be decomposed into union of two disjoint events in the following way:

$$B = (B \cap A) \cup (B \cap A^c),$$

and therefore

$$P(B) = P(B \cap A) + P(B \cap A^c).$$

Subtracting the second equality from the first one, we get:

$$P(A \cup B) - P(B) = P(A) - P(B \cap A),$$

from where we obtain the desired property.

□

3 Conditional Probability

Conditional probability is used to talk about the outcome of the experiment based on known partial information. The typical problems can be the following:

i). Assume that sum of two rolls of a fair die is equal to 9. What is the probability that on the first roll we got 6?

ii). How likely is it that the person has a disease given that a medical test was negative?

Thus, given the experiment, we know that the outcome is in some set B. We would like to quantify the probability that the outcome is also in A. This probability will be denoted by $P(A|B)$.

Example 3.1. *Consider roll of a fair die. Assume that it is given that the outcome is even. What is the probability that we got 6? There are 3 possible outcomes, in case the outcome is even: 2, 4, and 6. All of them are equally likely, and thus*

$$P(\textit{outcome is } 6|\textit{outcome is even}) = 1/3.$$

This example can give us an idea how to define the conditional probability. In discrete case,

$$P(A|B) = \frac{\text{number of elements in } A \cap B}{\text{number of elements in } B} \tag{3}$$

In general case, we have:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}. \tag{4}$$

So defined probability is indeed a probability law. We will check that it satisfies all the axioms of a probability law:

1. **(Nonnegativity)** Since both $P(A \cap B)$ and $P(B)$ are nonnegative, $P(A|B)$ is nonnegative.
2. **(Normalization)** $P(\Omega|B) = \dfrac{P(\Omega \cap B)}{P(B)} = \dfrac{P(B)}{P(B)} = 1.$
3. **(Additivity)** If A and C are disjoint, then:

$$P(A \cup C|B) = \frac{P((A \cup C) \cap B)}{P(B)} = \frac{P((A \cap B) \cup (C \cap B))}{P(B)} = \frac{P(A \cap B) + P(C \cap B)}{P(B)} = \\ = P(A|B) + P(C|B).$$

Since, so defined probability satisfies all the axioms of the probability law, all other properties of probability laws are satisfied for conditional probability, for example,

$$P(A \cup C|B) \leq P(A|B) + P(C|B).$$

Example 3.2. *Consider 3 tosses of a fair coin. Let* $A = \{\textit{more heads than tails}\}$*, and* $B = \{\textit{the 1st toss was a head}\}$*. Then* $B = \{HHH, HHT, HTH, HTT\}$ *and* $A \cap B = \{HHH, HHT, HTH\}$*. Since there is 8 possible outcomes in case of 3 tosses,* $P(B) = 4/8$ *and* $P(A \cap B) = 3/8$*. Therefore,* $P(A|B) = \dfrac{3/8}{4/8} = \dfrac{3}{4}$.

The formula for conditional probability can be rewritten in the way

$$P(A \cap B) = P(B)P(A|B). \tag{5}$$

Example 3.3. *If aircraft is present, radar detects it correctly with probability 0.99. If the aircraft is not present, the radar is mistaken (and tells that it's present) with probability 0.1. Assume that the probability that the aircraft is present is equal to 0.05. Determine the probabilities of false alarm (aircraft is not present, and radar detected it) and of missed detection (aircraft is present, but the radar tells that it's not).*

Let the event A be an event that aircraft is present, and event R be an event that the radar detected the aircraft. We are given that: $P(A) = 0.05$, $P(R|A) = 0.99$, and $P(R|A^c) = 0.1$. We need to find $P(R \cap A^c)$ – probability of false alarm, and $P(R^c \cap A)$ – probability of missed detection.

$$P(\textit{false alarm}) = P(R \cap A^c) = P(A^c)P(R|A^c) = 0.95 \cdot 0.1 = 0.095,$$

and

$$P(\textit{missed detection}) = P(R^c \cap A) = P(A)P(R^c|A) = 0.05 \cdot 0.01 = 0.0005.$$