AMS210.01.
Homework 5

Andrei Antonenko

Due at the beginning of the class, April 14

1. Determine the number of inversions and the sign of the following permutations:
   (a) (51423)
   (b) (54321)
   (c) $(n \;\; n-1 \;\; n-2 \;\; \dots \;\; 2 \;\; 1)$ (all numbers from 1 to n in the reverse order)
2. Check which of the following terms are included in the expression for the determinant (for 5×5-matrix) and with which signs.
   (a) $a_{15}a_{21}a_{35}a_{42}a_{53}$
   (b) $a_{13}a_{25}a_{31}a_{45}a_{54}$
   (c) $a_{22}a_{34}a_{41}a_{12}a_{55}$
3. Compute the following determinants:
   (a) $\begin{vmatrix} 3 & 5 \\ 5 & 3 \end{vmatrix}$
   (b) $\begin{vmatrix} ab & ac \\ bd & cd \end{vmatrix}$
4. Compute the following determinants:
   (a) $\begin{vmatrix} 1 & 2 & 3 \\ 5 & 1 & 4 \\ 3 & 2 & 5 \end{vmatrix}$
   (b) $\begin{vmatrix} 0 & a & 0 \\ b & c & d \\ 0 & e & 0 \end{vmatrix}$
5. Compute the following determinants:
   (a) $\begin{vmatrix} a_{11} & 0 & 0 & \dots & 0 \\ a_{21} & a_{22} & 0 & \dots & 0 \\ a_{31} & a_{32} & a_{33} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix}$

(b) $\begin{vmatrix} 0 & \dots & 0 & 0 & a_{1n} \\ 0 & \dots & 0 & a_{2,n-1} & a_{2n} \\ 0 & \dots & a_{3,n-2} & a_{3,n-1} & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & \dots & a_{n,n-2} & a_{n,n-1} & a_{nn} \end{vmatrix}$

6. How do the determinant of the matrix changes if

   (a) change the sign of all entries of the matrix.

   (b) to each of the rows add all the rows preceding it.

   (c) put the first row on the last place, and all other rows move up.

7. Compute the following determinants using elementary row operations.

   (a) $\begin{vmatrix} 1 & 10 & 100 & 1000 & 10000 & 100000 \\ 0.1 & 2 & 30 & 400 & 5000 & 60000 \\ 0 & 0.1 & 3 & 60 & 1000 & 15000 \\ 0 & 0 & 0.1 & 4 & 100 & 2000 \\ 0 & 0 & 0 & 0.1 & 5 & 150 \\ 0 & 0 & 0 & 0 & 0.1 & 6 \end{vmatrix}.$

   (b) $\begin{vmatrix} 1 & 2 & 3 & \dots & n \\ -1 & 0 & 3 & \dots & n \\ -1 & -2 & 0 & \dots & n \\ \dots & \dots & \dots & \dots & \dots \\ -1 & -2 & -3 & \dots & 0 \end{vmatrix}$

   (c) $\begin{vmatrix} 1 & n & n & \dots & n \\ n & 2 & n & \dots & n \\ n & n & 3 & \dots & n \\ \dots & \dots & \dots & \dots & \dots \\ n & n & n & \dots & n \end{vmatrix}$

   (d) $\begin{vmatrix} 1 & a_1 & a_2 & \dots & a_n \\ 1 & a_1+b_1 & a_2 & \dots & a_n \\ 1 & a_1 & a_2+b_2 & \dots & a_n \\ \dots & \dots & \dots & \dots & \dots \\ 1 & a_1 & a_2 & \dots & a_n+b_n \end{vmatrix}$

8. Using the expansion by the 3rd row, find the determinant:

$$\begin{vmatrix} 2 & -3 & 4 & 1 \\ 4 & -2 & 3 & 2 \\ a & b & c & d \\ 3 & -1 & 4 & 3 \end{vmatrix}$$

9. Compute the following determinants using expansion by a row (or column)

(a) $$\begin{vmatrix} x & y & 0 & \dots & 0 & 0 \\ 0 & x & y & \dots & 0 & 0 \\ 0 & 0 & x & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & x & y \\ y & 0 & 0 & \dots & 0 & x \end{vmatrix}$$

(b) $$\begin{vmatrix} a_0 & -1 & 0 & 0 & \dots & 0 & 0 \\ a_1 & x & -1 & 0 & \dots & 0 & 0 \\ a_2 & 0 & x & -1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n-1} & 0 & 0 & 0 & \dots & x & -1 \\ a_n & 0 & 0 & 0 & \dots & 0 & x \end{vmatrix}$$

10. Compute the following determinant by squaring the matrix.

$$\begin{vmatrix} a & b & c & d \\ -b & a & d & -c \\ -c & -d & a & b \\ -d & c & -b & a \end{vmatrix}$$

11. Solve the following systems by Cramer's rule

(a) $$\begin{cases} 2x_1 + 5x_2 = 1 \\ 3x_1 + 7x_2 = 2 \end{cases}$$

(b) $$\begin{cases} x_1 + x_2 + x_3 = 6 \\ -x_1 + x_2 + x_3 = 0 \\ x_1 - x_2 + x_3 = 2 \end{cases}$$

(c) $$\begin{cases} x_1 \cos\alpha + x_2 \sin\alpha = \cos\beta \\ -x_1 \sin\alpha + x_2 \cos\alpha = \sin\beta \end{cases}$$

12. **[Extra credit]** It is known that numbers 20604, 53227, 25755, 20927 and 289 can be divided by 17. Prove that the determinant

$$\begin{vmatrix} 2 & 0 & 6 & 0 & 4 \\ 5 & 3 & 2 & 2 & 7 \\ 2 & 5 & 7 & 5 & 5 \\ 2 & 0 & 9 & 2 & 7 \\ 0 & 0 & 2 & 8 & 9 \end{vmatrix}$$

divides by 17.

13. **[Extra credit]** Compute the following determinant (use properties!)

$$\begin{vmatrix} a_1 + x & a_2 & \dots & a_n \\ a_1 & a_2 + x & \dots & a_n \\ \dots & \dots & \dots & \dots \\ a_1 & a_2 & \dots & a_n + x \end{vmatrix}$$

14. **[Extra credit]** Compute the following determinant (use elementary row operations).

$$\begin{vmatrix} x_1 & a_{12} & a_{13} & \dots & a_{1n} \\ x_1 & x_2 & a_{23} & \dots & a_{2n} \\ x_1 & x_2 & x_3 & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ x_1 & x_2 & x_3 & \dots & x_n \end{vmatrix}$$

15. **[Extra credit]** Compute the following determinant (use elementary row operations).

$$\begin{vmatrix} 1 & 1 & \dots & 1 & -n \\ 1 & 1 & \dots & -n & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & -n & \dots & 1 & 1 \\ -n & 1 & \dots & 1 & 1 \end{vmatrix}$$

16. **[Extra credit]** Compute the following determinant (use expansion).

$$\begin{vmatrix} a_0 & 1 & 1 & 1 & \dots & 1 \\ 1 & a_1 & 0 & 0 & \dots & 0 \\ 1 & 0 & a_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & 0 & \dots & a_n \end{vmatrix}$$

17. **[Extra credit]** Compute the following determinant (use expansion).

$$\begin{vmatrix} a_1 & 0 & \dots & 0 & b_1 \\ 0 & a_2 & \dots & b_2 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & b_{2n-1} & \dots & a_{2n-1} & 0 \\ b_{2n} & 0 & \dots & 0 & a_{2n} \end{vmatrix}$$

18. **[Extra credit]** Compute the following determinant (use recurrency).

$$\begin{vmatrix} 0 & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & x & \dots & x & x \\ 1 & x & 0 & \dots & x & x \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & x & x & \dots & 0 & x \\ 1 & x & x & \dots & x & 0 \end{vmatrix}$$