AMS210.01.
Homework 3

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Due at the beginning of the class, March 14, 2003

1. Which of the following statements are true or false? Explain your answer. If false, give a counterexample.
   (a) For any u_1, u_2, and u_3 from V, the system $\mathbf{0}, u_1, u_2, u_3$ is linearly dependent.
   (b) If u_1, u_2, and u_3 span V, then u_1, u_2, u_3, and w span V for any $w \in V$.
   (c) If u_1, u_2, and u_3 span V, then $\dim V = 3$.
   (d) If u_1, u_2, and u_3 span V, then they are independent.
   (e) If u_1, u_2, and u_3 form a basis for V, then they are independent.
   (f) If u_1, u_2 are independent, then they form a basis of V.
   (g) If u_1, u_2, u_3, and u_4 are linearly independent, then $\dim V = 4$.
   (h) If u_1, u_2, and u_3 are linearly independent, then $\dim V \geq 3$.
2. Consider the vector space $\mathbb{R}^4$. Let $u_1 = (1, 2, -1, 0)$, $u_2 = (2, 1, 1, 1)$, and $u_3 = (-1, 0, 0, 1)$.
   (a) Find $2u_1 + u_2 - u_3$.
   (b) Find vector $x \in \mathbb{R}^4$ such that $2u_1 + u_2 + 2u_3 + x = \mathbf{0}$.
3. Consider the vector space $\mathbb{R}^3$. Let $u_1 = (1, 2, 1)$, $u_2 = (2, 1, 3)$, and $v = (1, 2, \lambda)$, $\lambda \in \mathbb{R}$. Find all λ's such that the vector v can be expressed as a linear combination of u_i's.
4. Determine which of the following sets with standard operations form a vector space (over the field $\mathbb{R}$). Explain your answers.
   (a) $\{(x, y, x) | x, y \in \mathbb{R}\}$
   (b) $\{(0, x, y) | x, y \in \mathbb{R}\}$
   (c) $\{(x, y, 1) | x, y \in \mathbb{R}\}$
   (d) $\{(x, y, x + y) | x, y \in \mathbb{R}\}$
   (e) $\{(x, y) | x, y \in \mathbb{R}, x \geq 0\}$
   (f) Set of all polynomials of even power: $\left\{f(t) = \sum_{i=0}^{n} a_i t^i | a_i \in \mathbb{R}, n \text{ is even}\right\}$
   (g) Set of all symmetrical 2×2-matrices: $\left\{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} | a, b, c \in \mathbb{R} \right\}$

(h) Set of all diagonal $n \times n$-matrices.

(i) The following set of matrices: $\left\{ \begin{pmatrix} a & b & 2a \\ 2d & c & d \end{pmatrix} | a, b, c, d \in \mathbb{R} \right\}$

(j) $\mathbb{Z}^n = \{(x_1, \dots, x_n) | x_i \in \mathbb{Z}\}$

5. Let's consider the following set $\{(x, y) | x, y \in \mathbb{R}\}$ and define operations of addition and multiplication as follows:

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 + y_2), \quad x_i, y_i \in \mathbb{R}$$
$$c \odot (x, y) = (cx, y), \quad c, x, y \in \mathbb{R}$$

Prove that this is not a vector space (over a field $\mathbb{R}$)? Which axioms of vector space are not satisfied?

6. Determine whether vector v belongs to the span of vectors u_i's (i.e., can v be expressed as a linear combination of u_i's?). Explain your answer. If the answer is yes, find the coefficients of linear combination.

(a) Space: $\mathbb{R}^3$. Vectors: $v = (6, 9, 14)$, $u_1 = (1, 1, 1)$, $u_2 = (1, 1, 2)$, and $u_3 = (1, 2, 3)$.

(b) Space: $\mathbb{P}_2$. Vectors: $v = t^2 + 2t + 2$, $u_1 = t^2 + 1$, $u_2 = t^2 - t$, and $u_3 = t^2 + t + 2$.

(c) Space: $\mathbb{R}^4$. Vectors: $v = (0, 1, 1, 0)$, $u_1 = (1, 0, 0, 1)$, $u_2 = (1, -1, 0, 0)$, and $u_3 = (0, 1, 2, 1)$.

(d) Space: $M_{2,2}$. Vectors: $v = \begin{pmatrix} 5 & 1 \\ -1 & 9 \end{pmatrix}$, $u_1 = \begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix}$, $u_2 = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$, and $u_3 = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}$.

(e) Space: $\mathbb{P}_2$. Vectors: $v = t + 1$, $u_1 = t^2 - t$, $u_2 = t^2 - 2t + 1$, and $u_3 = -t^2 + 1$.

(f) Space: $\mathbb{R}^3$. Vectors: $v = (2, -1, -1)$, $u_1 = (1, -1, 0)$, $u_2 = (1, -2, 1)$, and $u_3 = (-1, 0, 1)$.

7. Does the following systems of vectors span V? Explain your answers.

(a) Space: $V = \mathbb{R}^3$. Vectors: $u_1 = (1, 1, 0)$, $u_2 = (1, 0, 1)$, and $u_3 = (0, -1, 1)$.

(b) Space: $V = \mathbb{R}^3$. Vectors: $u_1 = (1, 1, 1)$, $u_2 = (1, 0, 1)$, and $u_3 = (0, -1, 1)$.

(c) Space: $V = \mathbb{P}_2$. Vectors: $u_1 = 2t^2 + 2t + 1$, $u_2 = t^2 + 1$, $u_3 = t + 1$.

(d) Space: $V = M_{2,2}$. Vectors: $u_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $u_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, $u_3 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, and $u_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

8. Determine whether the following systems of vectors are linearly dependent or independent. Explain your answer.

(a) Space: $\mathbb{R}^3$. Vectors: $u_1 = (1, 2, 1)$, $u_2 = (-1, 0, 1)$, and $u_3 = (3, 8, 5)$.

(b) Space: $\mathbb{R}^3$. Vectors: $u_1 = (1, 2, 1)$, $u_2 = (-1, 0, 1)$, and $u_3 = (0, 1, 0)$.

(c) Space: $\mathbb{R}^2$. Vectors: $u_1 = (1, 2)$, $u_2 = (2, 5)$, and $u_3 = (7, 7)$.

(d) Space: $\mathbb{R}^3$. Vectors: $u_1 = (2, 5, 6)$, $u_2 = (0, 0, 0)$, and $u_3 = (7, 8, 9)$.

(e) Space: $\mathbb{P}_2$. Vectors: $u_1 = t^2 + 1$, $u_2 = 2t^2 - 2$, and $u_3 = 1$.

(f) Space: $M_{2,2}$. Vectors: $u_1 = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$, $u_2 = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}$, $u_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and $u_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

9. Does the following sets of vectors form a basis for the specified vector spaces? Explain your answer.

   (a) Space: $\mathbb{R}^3$. Vectors: $u_1 = (1, 2, 3)$, $u_2 = (1, 3, 2)$.

   (b) Space: $\mathbb{R}^2$. Vectors: $u_1 = (1, 1)$, $u_2 = (2, -5)$, $u_3 = (4, 3)$.

   (c) Space: $\mathbb{R}^3$. Vectors: $u_1 = (1, 0, 1)$, $u_2 = (0, 1, 0)$, $u_3 = (-1, 0, 1)$.

   (d) Space: $\mathbb{R}^3$. Vectors: $u_1 = (2, 1, -3)$, $u_2 = (3, 2, -5)$, $u_3 = (1, -1, 1)$.

   (e) Space: $\mathbb{P}_2$. Vectors: $u_1 = t^2 + t$, $u_2 = t + 2$, $u_3 = t^2 + t + 2$.

10. (a) Find the basis and determine the dimension of the vector space of symmetric 2×2-matrices.

    (b) Find the basis and determine the dimension of the vector space of symmetric $n \times n$-matrices.

11. (a) Find the basis and determine the dimension of the vector space of skew-symmetric 2×2-matrices.

    (b) Find the basis and determine the dimension of the vector space of skew-symmetric $n \times n$-matrices.

12. Find the basis and determine the dimension of the vector space of diagonal $n \times n$-matrices.

13. (a) Find the basis of $\mathbb{R}^3$ containing the following vector $u_1 = (2, 3, 0)$.

    (b) Find the basis of $\mathbb{P}_3$ containing the following 2 vectors $u_1(t) = t^3 + t^2$, $u_2(t) = t^2 + t + 1$.

14. Find the dimension and basis of the span of the following vectors in the specified vector space.

    (a) Space: $\mathbb{R}^3$. Vectors: $u_1 = (1, 3, 5)$, $u_2 = (2, 2, 2)$, $u_3 = (0, 1, 0)$.

    (b) Space: $\mathbb{R}^3$. Vectors: $u_1 = (1, 3, 1)$, $u_2 = (2, 2, 2)$, $u_3 = (4, 1, 4)$.

    (c) Space: $\mathbb{P}_3$. Vectors: $u_1 = 3t^2 + 2$, $u_2 = t^2 + 4t + 2$, $u_3 = 5t^2 - 4t + 2$.

    (d) Space: $\mathbb{R}^3$. Vectors: $u_1 = (2, 1, 3)$, $u_2 = (1, 2, 3)$, $u_3 = (6, 6, 12)$, $u_4(3, 3, 6)$, $u_5 = (1, -1, 0)$, $u_6 = (4, 5, 9)$.

    (e) Space: $\mathbb{R}^4$. Vectors: $u_1 = (1, 3, 2, 1)$, $u_2 = (2, -1, 1, -2)$, $u_3 = (0, 7, 3, 4)$, $u_4 = (3, 2, 3, -1)$.

    (f) Space: $\mathbb{P}_3$. Vectors: $u_1 = t^3 + t^2 - 2t + 1$, $u_2 = t^2 + 1$, $u_3 = t^3 - 2t$, $u_4 = 2t^3 + 3t^2 - 4t + 3$.

15. Find the ranks of the following matrices

    (a) $\begin{pmatrix} 1 & 7 & 4 & -2 & 5 \\ 8 & 2 & 2 & -1 & 1 \\ -2 & 4 & 2 & -1 & 3 \end{pmatrix}$

    (b) $\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}$

16. For all values of $\lambda \in \mathbb{R}$ find the rank of the following matrix:

$$\begin{pmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & 1-\lambda & 0 & 0 \\ 0 & 0 & 2-\lambda & 3 \\ 0 & 0 & 0 & 3-\lambda \end{pmatrix}$$

17. **[Extra credit]** Consider the set of all positive real numbers $\mathbb{R}_+$. Define operations as follows:

$$a \oplus b = ab, \quad a, b \in \mathbb{R}_+$$
$$c \odot a = a^c, \quad a \in \mathbb{R}_+, \quad c \in \mathbb{R}$$

    Is it a vector space (over a field $\mathbb{R}$)?

18. **[Extra credit]** Prove that any $n \times n$-matrix can be represented as a sum of symmetric matrix and skew-symmetric matrix.

19. **[Extra credit]** For all values of λ find the rank of the following matrix:

$$\begin{pmatrix} \lambda & 1 & 2 & \dots & n-1 & 1 \\ 1 & \lambda & 2 & \dots & n-1 & 1 \\ 1 & 2 & \lambda & \dots & n-1 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 2 & 3 & \dots & \lambda & 1 \\ 1 & 2 & 3 & \dots & n & 1 \end{pmatrix}$$