Lecture 28

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1 Operators

In this lecture we will start studying the most important part of the course on linear algebra — the theory of operators.

Let V be a vector space. Any linear function from V to V is called the **linear operator**. We will denote operators by script letters ($\mathcal{A}, \mathcal{B}, \mathcal{C}$), for example:

$$\mathcal{A} : V \to V.$$

It means that the operator "rearranges" somehow vectors from the given vector space V. An example of the operator can be an operator of rotation by an angle α — applying it, each vector will be rotated by an angle α counterclockwise. Another example — operator of the reflection with respect to given line — each vector maps to its reflection with respect to some line.

We will learn how to describe operators, and then we will try to classify them, and understand actions of them. But first we will study some more facts about vector spaces.

2 Coordinates

Let V be a vector space, and assume that $\{e_1, e_2, \ldots, e_n\}$ is a basis. Since it is a basis, we can represent any other vector v as a linear combination of basic vectors, i.e. for any vector v we can find such numbers $a_1, a_2, \ldots, a_n$, that

$$v = a_1e_1 + a_2e_2 + \cdots + a_ne_n.$$

Such numbers $a_1, a_2, \ldots, a_n$ are called **coordinates** of the vector v with respect to basis $\{e_1, e_2, \ldots, e_n\}$. Let's consider some examples.

Example 2.1. *Consider the space $\mathbb{R}^3$, and let vector $v = (1, 1, 1)$. We will consider 2 different bases, and find coordinates of v with respect to them*

- *Let's consider the standard basis*

$$e_1 = (1, 0, 0), \quad e_2 = (0, 1, 0), \quad e_3 = (0, 0, 1).$$

Then the vector v can be represented as

$$v = (1, 1, 1) = 1(1, 0, 0) + 1(0, 1, 0) + 1(0, 0, 1) = 1e_1 + 1e_2 + 1e_3.$$

So, the coordinates of this vector with respect to the standard basis are $(1, 1, 1)$.

- *Let's consider another basis*

$$e'_1 = (0, 0, 1), \quad e'_2 = (0, 1, 1), \quad e'_3 = (1, 1, 1)$$

(we can simply check that this is a basis). Then the vector v can be represented as

$$v = (1, 1, 1) = 0(0, 0, 1) + 0(0, 1, 1) + 1(1, 1, 1) = 0e_1 + 0e_2 + 1e_3.$$

So, the coordinates of this vector with respect to the basis $\{e'_1, e'_2, e'_3\}$ *are* $(0, 0, 1)$.

So, we see, that the coordinates of the same vector may be different with the respect to different bases.

Now we will prove that if the basis is given, the coordinates are defined uniquely.

Theorem 2.2. *Let V be a vector space, and let $e_1, e_2, \ldots, e_n$ be a basis. Then for any vector v numbers $a_1, a_2, \ldots, a_n$ such that*

$$v = a_1e_1 + a_2e_2 + \cdots + a_ne_n$$

are defined uniquely, i.e. if

$$v = a_1e_1 + a_2e_2 + \cdots + a_ne_n = b_1e_1 + b_2e_2 + \cdots + b_ne_n$$

then $a_1 = b_1, a_2 = b_2, \ldots, a_n = b_n$.

Proof. If

$$v = a_1e_1 + a_2e_2 + \cdots + a_ne_n = b_1e_1 + b_2e_2 + \cdots + b_ne_n$$

then

$$\begin{aligned}\mathbf{0} &= (a_1e_1 + a_2e_2 + \cdots + a_ne_n) - (b_1e_1 + b_2e_2 + \cdots + b_ne_n) \\ &= (a_1 - b_1)e_1 + (a_2 - b_2)e_2 + \cdots + (a_n - b_n)e_n.\end{aligned}$$

But vectors $e_1, e_2, \ldots, e_n$ are linearly independent, thus coefficients of this linear combination are equal to 0, i.e.

$$a_1 = b_1, \quad a_2 = b_2, \quad \ldots, \quad a_n = b_n.$$

□

3 Change of the basis

Let V be a vector space, and assume that there are 2 fixed bases in V:

$$\{e_1, e_2, \dots, e_n\} \text{ — “old” basis}$$
$$\{e'_1, e'_2, \dots, e'_n\} \text{ — “new” basis}$$

Let's consider any vector v. Then we can find its coordinates with respect to the “old” basis and its coordinates with respect to the “new” basis. Our goal is to figure out how are they related.

Let's do the following construction. Since $\{e_1, e_2, \dots, e_n\}$ is a basis, we can express all vectors from the “new” basis as linear combinations of the vectors from the “old” basis.

$$e'_1 = c_{11}e_1 + c_{21}e_2 + \dots + c_{n1}e_n$$
$$e'_2 = c_{12}e_1 + c_{22}e_2 + \dots + c_{n2}e_n$$
$$\dots$$
$$e'_2 = c_{1n}e_1 + c_{2n}e_2 + \dots + c_{nn}e_n$$

Now we can write the coefficients of these expressions as columns of the matrix:

$$C = \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \dots & \dots & \dots & \dots \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{pmatrix}$$

This matrix is called **change-of-basis matrix** from the “old” basis $\{e_1, e_2, \dots, e_n\}$ to the “new” basis $\{e'_1, e'_2, \dots, e'_n\}$.

Straight from the definition of the change-of-basis matrix it follows, that

$$(e'_1, e'_2, \dots, e'_n) = (e_1, e_2, \dots, e_n) \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \dots & \dots & \dots & \dots \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{pmatrix}$$

i.e.

$$(e'_1, e'_2, \dots, e'_n) = (e_1, e_2, \dots, e_n)C \tag{1}$$

Example 3.1. *Let's consider $\mathbb{R}^3$. Assume that the “old” basis is*

$$e_1 = (1, 0, 1), \quad e_2 = (2, -1, 0), \quad e_3 = (0, 1, 3),$$

and the “new” basis is

$$e'_1 = (3, 0, 4), \quad e'_2 = (4, -1, 2), \quad e'_3 = (-3, 2, 2),$$

Let's express vectors from the "new" basis as linear combinations of the vectors from the "old" basis.

$\boldsymbol{e_1'}$:

$$e_1' = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} = c_{11} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_{21} \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + c_{31} \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$$

So, we have the following system:

$$\begin{cases} c_{11} + 2c_{21} = 3 \\ -c_{21} + c_{31} = 0 \\ c_{11} + 3c_{31} = 4 \end{cases}$$

Solving this system, we can obtain $c_{11} = 1, c_{21} = 1, c_{31} = 1$. *So,*

$$e_1' = e_1 + e_2 + e_3.$$

$\boldsymbol{e_2'}$:

$$e_2' = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} = c_{12} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_{22} \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + c_{32} \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$$

So, we have the following system:

$$\begin{cases} c_{12} + 2c_{22} = 4 \\ -c_{22} + c_{32} = -1 \\ c_{12} + 3c_{32} = 2 \end{cases}$$

Solving this system, we can obtain $c_{12} = 2, c_{22} = 1, c_{32} = 0$. *So,*

$$e_2' = 2e_1 + e_2.$$

$\boldsymbol{e_3'}$:

$$e_3' = \begin{pmatrix} -3 \\ 2 \\ 2 \end{pmatrix} = c_{13} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_{23} \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + c_{33} \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$$

So, we have the following system:

$$\begin{cases} c_{13} + 2c_{23} = -3 \\ -c_{23} + c_{33} = 2 \\ c_{13} + 3c_{33} = 2 \end{cases}$$

Solving this system, we can obtain $c_{13} = -1, c_{23} = -1, c_{33} = 1$. *So,*

$$e_3' = -e_1 - e_2 + e_3.$$

Writing all these coordinates as a columns of a matrix, we will obtain the change-of-coordinates matrix:

$$C = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}$$

from the "old" basis $\{e_1, e_2, e_3\}$ *to the "new" basis* $\{e'_1, e'_2, e'_3\}$.

Now we will learn what do we need this matrix for.

Let x be a vector from the vector space V with 2 bases — "old" and "new". Then we can represent as a linear combination of vectors from the "old" basis and as a linear combination of vectors from the "new" basis:

$$x = x_1e_1 + x_2e_2 + \cdots + x_ne_n = x'_1e'_1 + x'_2e'_2 + \cdots + x'_ne'_n$$

In the matrix form we can write it as

$$x = (e_1, e_2, \ldots, e_n) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = (e'_1, e'_2, \ldots, e'_n) \begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{pmatrix}$$

Now, using formula (1) we can substitute $(e'_1, e'_2, \ldots, e'_n)$ and get the following:

$$x = (e_1, e_2, \ldots, e_n) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = (e_1, e_2, \ldots, e_n)C \begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{pmatrix}$$

So, now we see the following important result:

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = C \begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{pmatrix}, \quad \text{or} \quad C^{-1} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{pmatrix} \tag{2}$$

So, given the coordinates of the vector with respect to the "new" basis and the change-of-basis matrix we can get the coordinates of the vector with respect to the "old" basis, and the other way round.

Example 3.2 (Continuation of the example 3.1). *Let the bases are the same as in the previous example, i.e. the "old" basis is*

$$e_1 = (1, 0, 1), \quad e_2 = (2, -1, 0), \quad e_3 = (0, 1, 3),$$

and the "new" basis is

$$e_1' = (3,0,4), \quad e_2' = (4,-1,2), \quad e_3' = (-3,2,2),$$

As we computed in the previous example, the change-of-basis matrix is

$$C = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}$$

Let $x = (-2,4,10)$. Let's try to find the coordinates of this vector with respect to the "old" and to the "new" bases, and see that they are connected be relation (2).

For the coordinates with respect to the "old" basis, we should solve the following system:

$$x = \begin{pmatrix} -2 \\ 4 \\ 10 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$$

Solving it, e.g using Gaussian elimination, we get

$$x_1 = -2, \quad x_2 = 0, \quad x_3 = 4.$$

For the coordinates with respect to the "new" basis, we should solve the following system:

$$x = \begin{pmatrix} -2 \\ 4 \\ 10 \end{pmatrix} = x_1' \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} + x_2' \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} + x_3' \begin{pmatrix} -3 \\ 2 \\ 2 \end{pmatrix}$$

Solving it, e.g. using Gaussian elimination, we get

$$x_1' = 3, \quad x_2' = -2, \quad x_3' = 1.$$

Now we can check our main formula (2):

$$C \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

4 Matrix of the linear operator

Let V be a vector space, and let $\{e_1, e_2, \dots, e_n\}$ be the basis of V. Let $\mathcal{A}$ be a linear operator in the space V.

Let's apply this linear operator to basic vectors, and let's represent them as a linear combination of vectors from the basis, i.e.

Represent $\mathcal{A}(e_1)$ as a linear combination of $e_1, e_2, \dots, e_n$

Represent $\mathcal{A}(e_2)$ as a linear combination of $e_1, e_2, \dots, e_n$

$\dots$

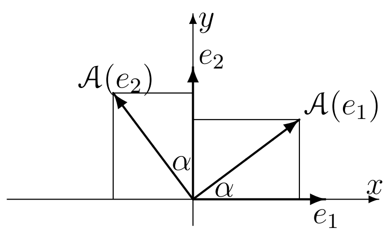
Represent $\mathcal{A}(e_n)$ as a linear combination of $e_1, e_2, \dots, e_n$

Now, we will write the corresponding coordinates as columns of a matrix: coordinates of $\mathcal{A}(e_1)$ to the first column, coordinates of $\mathcal{A}(e_2)$ to the second column, etc. This matrix will be called the **matrix of a linear operator**.

Actually, it means that if the operator $\mathcal{A}$ has a matrix A then

$$(\mathcal{A}(e_1), \mathcal{A}(e_2), \ldots, \mathcal{A}(e_n)) = (e_1, e_2, \ldots, e_n)A \tag{3}$$

Example 4.1 (Matrix of rotation by an angle α). *We will consider the operator of the rotation of the plane by an angle α counterclockwise.*



From this picture we can see that the coordinates of the vector $\mathcal{A}(e_1)$ are $(\cos\alpha, \sin\alpha)$, and coordinates of the vector $\mathcal{A}(e_2)$ are $(-\sin\alpha, \cos\alpha)$. So, the matrix of the operator of rotation of the plane by an angle α is

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

As for linear functions, the following fact is true: if

$$y = \mathcal{A}(x)$$

then

$$y = Ax.$$

Example 4.2. *Let's consider a vector $(1, 1)$. Let's find its coordinates after rotation by an angle α counterclockwise. To do this we should multiply the matrix of rotation by the column with coordinates of the vector. We get:*

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\alpha - \sin\alpha \\ \cos\alpha + \sin\alpha \end{pmatrix}.$$

So, after rotation by an angle α counterclockwise the vector $(1, 1)$ will become $(\cos\alpha - \sin\alpha, \cos\alpha + \sin\alpha)$.