Lecture 26

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1 Orthogonality

Let V be a Euclidean space, and let v and u be 2 vectors in this space. Then we can define the angle between these 2 vectors.

Definition 1.1. *The* ***angle*** θ *between two vectors* v *and* u *from the vector space* V *can be defined by the following formula:*

$$\cos\theta = \frac{\langle u, v\rangle}{\|u\|\|v\|}$$

Actually, we should check that this definition is correct — we have an expression for cosine, and it should belong to the interval $[-1, 1]$! It is easy to check. From Cauchy-Bunyakovsky-Schwartz inequality we have

$$|\langle u, v\rangle| \leq \|u\|\|v\|,$$

and so we will have

$$-1 \leq \frac{\langle u, v\rangle}{\|u\|\|v\|} \leq 1$$

Let's note that this is a general definition which works in many different spaces, and not only in $\mathbb{R}^2$ and $\mathbb{R}^3$. For example, by this formula we can define the angle between two functions from $C[a, b]$ or between two polynomials.

Example 1.2. *Let* $u = (1, 2, 3)$ *and let* $v = (-1, 2, -2)$. *Then* $\langle u, v\rangle = -1 + 4 - 6 = -4$, $\|u\| = \sqrt{1 + 4 + 9} = \sqrt{14}$, *and* $\|v\| = \sqrt{1 + 4 + 4} = 3$. *So, the angle* θ *between these two vectors can be defined by the following formula:*

$$\cos\theta = \frac{\langle u, v\rangle}{\|u\|\|v\|} = \frac{-4}{3\sqrt{14}}.$$

Another important concept is a normalization of the vector.

Definition 1.3. *If* v *is a vector from the vector space* V *then the vector*

$$\frac{v}{\|v\|}$$

is called a ***normalization*** *of* v.

The main property of normalization is that it's norm is equal to 1. So, we take a vector which is proportional to v with the "length" 1.

Now we'll give the very important definition — main definition of this lecture.

Definition 1.4. *Two vectors u and v are called **orthogonal** if*

$$\langle u, v\rangle = 0.$$

Let we have a vector v from the Euclidean space V. Let's consider all vectors u which are orthogonal to v, i.e. the set of vectors u such that $\langle v, u\rangle = 0$:

$$v^{\perp} = \{u \in V | \langle u, v\rangle = 0\} \quad \text{(read "}v\text{-perp")}$$

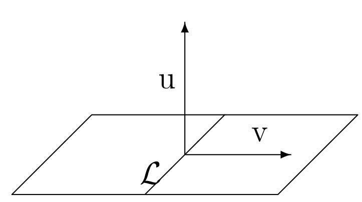
This set is called the **orthogonal complement** to the vector v. Now let S be a set of vectors. Then we can define $S^{\perp}$ as the following set:

$$S^{\perp} = \{u \in V | \langle u, v\rangle = 0 \text{ for all } v \in S\}$$

This set $S^{\perp}$ is called the **orthogonal complement** to the set S. Let's note that $S^{\perp}$ is a vector space. First, $\mathbf{0}$ is orthogonal to any other vector, since $\langle 0, v\rangle = 0$ for all v. So, $\mathbf{0} \in S^{\perp}$. Now, let $u_1, u_2 \in S^{\perp}$, such that $\langle u_1, v\rangle = 0$ for all v and $\langle u_2, v\rangle = 0$ for all v. So it follows that $\langle u_1 + u_2, v\rangle = \langle u_1, v\rangle + \langle u_2, v\rangle = 0$, and thus $u_1 + u_2$ belongs to $S^{\perp}$. Moreover, it is obvious to check that if u belongs to $S^{\perp}$ then ku belongs to it. Thus, we proved that $S^{\perp}$ is a vector space.

Geometrically speaking, let's consider the case of 3-dimensional space. Let v be a vector from the 3-dimensional space, then $v^{\perp}$ is the plane which is perpendicular to this vector.

Now let's consider more difficult case, when S consists of 2 vectors v and u. In this case the line $\mathcal{L}$ will be orthogonal to both vectors. It is illustrated on the picture below.



Our goal is to describe $u^{\perp}$ or $S^{\perp}$ somehow, for example give a basis of it. Actually, for the simple spaces, like $\mathbb{R}^n$ the basis of the orthogonal complement can be found as a basis in the solution space of the corresponding homogeneous system. We'll show it on the example.

Example 1.5. *Let $v = (1, -3, 4)$. Let's find the basis of the orthogonal complement to u, i.e. the basis of $u^{\perp}$. Vector $u = (x, y, z)$ is orthogonal to v if $\langle v, u\rangle = x - 3y + 4z = 0$. So, we have an equation:*

$$x - 3y + 4z = 0.$$

This can be considered as a homogeneous system with one equation and 3 variables. Here, variable x is leading and y, z are free. So, the basis can be obtained by assigning the value 1 to y and 0 to z, and after that 0 to y and 1 to z:

- $y = 1$; $z = 0$: $x = 3$, *so the corresponding vector is* $(3, 1, 0)$;
- $y = 0$; $z = 1$: $x = -4$, *so the corresponding vector is* $(-4, 0, 1)$.

Thus, the basis of the plane orthogonal to the vector $(1, -3, 4)$ *is* $\{(3, 1, 0); (-4, 0, 1)\}$.

Example 1.6. *Let* $S = \{(1, 2, 2), (2, 3, 1)\}$. *Let's find the basis of* $S^{\perp}$. *Vector* $v = (x, y, z)$ *is in the* $S^{\perp}$ *if* $\langle v, (1, 2, 2)\rangle = 0$ *and* $\langle v, (2, 3, 1)\rangle = 0$. *We can write it as a homogeneous system with 3 unknowns and 2 equations:*

$$\begin{cases} x + 2y + 2z = 0 \\ 2x + 3y + z = 0 \end{cases}$$

Reducing it to row echelon form we get the following system:

$$\begin{cases} x + 2y + 2z = 0 \\ \quad\; - y - 3z = 0 \end{cases}$$

Here, z is free variable and x and y are leading. So, The basis will consist of 1 vector and can be obtained by assigning the value 1 to z:

- $z = 1$: $y = -3$, $x = 4$, *so the corresponding vector is* $(1, -3, 4)$.

Thus, $S^{\perp}$ *is the line which contains the vector* $(1, -3, 4)$.

Now we will see why orthogonal vectors are nice. We will call vectors $v_1, v_2, \ldots, v_n$ orthogonal if each pair of them is orthogonal, i.e. if

$$\langle v_i, v_j\rangle = 0 \text{ for } i \neq j.$$

The following theorem gives an important property of orthogonal vectors.

Theorem 1.7. *If nonzero vectors $v_1, v_2 \ldots, v_n$ are orthogonal, then they are linearly independent.*

Proof. Let's write a zero linear combination of these vectors:

$$a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0.$$

Now, we'll multiply it by v_1. We will have:

$$\langle a_1 v_1 + a_2 v_2 + \cdots + a_n v_n, v_1\rangle = \langle 0, v_1\rangle = 0.$$

Using linearity of the scalar product we get:

$$a_1\langle v_1, v_1\rangle + a_2\langle v_2, v_1\rangle + \cdots + a_n\langle v_n, v_1\rangle = 0.$$

All terms except the first one are equal to 0, so we get

$$a_1\langle v_1, v_1\rangle = 0.$$

Since $\langle v_1, v_1\rangle \neq 0$ then $a_1 = 0$. In the same way we can prove that $a_2, a_3, \ldots, a_n = 0$. So, vectors are linearly independent. □

Another theorem gives us a mathematically exact formulation of the fact known from the school geometry.

Theorem 1.8 (Pythagoras theorem). *If u and v are orthogonal vectors, then*

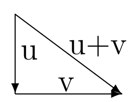
$$\|u + v\|^2 = \|u\|^2 + \|v\|^2.$$

Proof. We have:

$$\begin{aligned} \|u + v\|^2 &= \langle u + v, u + v\rangle \\ &= \langle u, u\rangle + \langle u, v\rangle + \langle v, u\rangle + \langle v, v\rangle \\ &= \langle u, u\rangle + 2\langle u, v\rangle + \langle v, v\rangle. \end{aligned}$$

Since v and u are orthogonal, $\langle u, v\rangle = 0$, and so $\|u + v\|^2 = \langle u, u\rangle + \langle v, v\rangle = \|u\|^2 + \|v\|^2$. □

In 2-dimensional space this theorem is exactly the Pythagoras theorem from the school geometry. The following picture illustrates it:



From this theorem it follows that if we have n orthogonal vectors in the n-dimensional vector space, then they form a basis. Moreover, this basis is nice, and we can find coordinates in this basis very easily. We'll demonstrate it in the next example.

Example 1.9. *Let $v_1 = (1, 2, 1)$, $v_2 = (2, 1, -4)$ and $v_3 = (3, -2, 1)$. We can check that these vectors are orthogonal, i.e. $\langle v_1, v_2\rangle = 0$, $\langle v_1, v_3\rangle = 0$, and $\langle v_2, v_3\rangle = 0$. So, they are linearly independent by theorem (1.7), and since there are 3 vectors, they form a basis in $\mathbb{R}^3$, i.e. each vector can be represented as a linear combination of them.*

Now let $u = (7, 1, 9)$. We want to represent u as a linear combination of v_1, v_2, and v_3, i.e. find coefficients a_1, a_2, and a_3 such that

$$u = a_1 v_1 + a_2 v_2 + a_3 v_3, \tag{1}$$

i.e.

$$(7,1,9) = a_1(1,2,1) + a_2(2,1,-4) + a_3(3,-2,1).$$

The familiar way to do it is to set up a linear system and solve it for unknowns a_1*,* a_2 *and* a_3.

But there is an easier way to do it. Let's multiply the equality (1) by v_1*,* v_2*, and then by* v_3*. We will have:*

$$\begin{aligned}\langle u, v_1\rangle &= a_1\langle v_1, v_1\rangle + a_2\langle v_2, v_1\rangle + \langle v_3, v_1\rangle \\ &= a_1\langle v_1, v_1\rangle,\end{aligned}$$

since $\langle v_2, v_1\rangle = \langle v_3, v_1\rangle = 0$*. So,*

$$a_1 = \frac{\langle u, v_1\rangle}{\langle v_1, v_1\rangle} = \frac{7+2+9}{1+4+1} = \frac{18}{6} = 3.$$

Now,

$$\begin{aligned}\langle u, v_2\rangle &= a_1\langle v_1, v_2\rangle + a_2\langle v_2, v_2\rangle + \langle v_3, v_2\rangle \\ &= a_2\langle v_2, v_2\rangle,\end{aligned}$$

since $\langle v_1, v_2\rangle = \langle v_3, v_2\rangle = 0$*. So,*

$$a_2 = \frac{\langle u, v_2\rangle}{\langle v_2, v_2\rangle} = \frac{14+1-36}{4+1+16} = \frac{-21}{21} = -1.$$

Now,

$$\begin{aligned}\langle u, v_3\rangle &= a_1\langle v_1, v_3\rangle + a_2\langle v_2, v_3\rangle + \langle v_3, v_3\rangle \\ &= a_3\langle v_3, v_3\rangle,\end{aligned}$$

since $\langle v_1, v_3\rangle = \langle v_2, v_3\rangle = 0$*. So,*

$$a_3 = \frac{\langle u, v_3\rangle}{\langle v_3, v_3\rangle} = \frac{21-2+9}{9+4+1} = \frac{28}{14} = 2.$$

So, we got that

$$u = 3v_1 - v_2 + 2v_3.$$