Lecture 25

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1 Euclidean spaces

Definition 1.1. *Let V be a vector space. Suppose to any 2 vectors $v, u \in V$ there assigned a number from $\mathbb{R}$ which will be denoted by $\langle v, u\rangle$ such that the following 3 properties hold:*

Bilinearity
- $\langle au_1 + bu_2, v\rangle = a\langle u_1, v\rangle + b\langle u_2, v\rangle$
- $\langle u, av_1 + bv_2\rangle = a\langle u, v_1\rangle + b\langle u, v_2\rangle$

Reflexivity $\langle u, v\rangle = \langle v, u\rangle$

Positivity $\langle u, u\rangle \geq 0$*; moreover if* $\langle u, u\rangle = 0$*, then* $u = 0$.

Any function which satisfy properties above is called a ***scalar (inner) product****. A vector space V with a scalar product is called a* ***(real) Euclidean space****.*

Now we will give popular examples of the scalar products in different spaces.

$\mathbb{R}^n$ The scalar product of 2 vectors x and y from $\mathbb{R}^n$ can be defined as following: if

$$x = (x_1, x_2, \ldots, x_n) \text{ and } y = (y_1, y_2, \ldots, y_n)$$

then

$$\langle x, y\rangle = x_1y_2 + x_2y_2 + \cdots + x_ny_n.$$

If the vectors are represented by column-vectors, i.e. by $n \times 1$-matrices, then

$$\langle x, y\rangle = x^\top y.$$

Example 1.2. *Let $x = (1, 2, 3)$ and $y = (3, -1, 4)$. Then $\langle x, y\rangle = 1\cdot 3 + 2\cdot(-1) + 3\cdot 4 = 13$.*

Actually, there are other ways of defining a scalar product in the vector space $\mathbb{R}^n$. For example, given positive numbers $a_1, a_2, \ldots, a_n$ we can define the scalar product to be

$$\langle (x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)\rangle = a_1(x_1y_1) + a_2(x_2y_2) + \cdots + a_n(x_ny_n).$$

$\boldsymbol{M_{m,n}}$ The scalar product of 2 $m \times n$-matrices A and B such that

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \text{ and } B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{pmatrix}$$

is equal to the sum of products of the corresponding entries:

$$\begin{aligned} \langle A, B \rangle = {} & a_{11}b_{11} + a_{12}b_{12} + \dots + a_{1n}b_{1n} \\ & + a_{21}b_{21} + a_{22}b_{22} + \dots + a_{2n}b_{2n} \\ & + \dots \\ & + a_{m1}b_{m1} + a_{m2}b_{m2} + \dots + a_{mn}b_{mn}. \end{aligned}$$

We can write this scalar product in terms of matrix multiplication, but to do it we need one more definition.

Definition 1.3. *The* ***trace*** *of the square matrix A is the sum of its diagonal elements. It is denoted by* $\operatorname{tr} A$.

Example 1.4.

$$\operatorname{tr} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = 1 + 5 + 9 = 15.$$

Now, using trace we can write that

$$\langle A, B \rangle = \operatorname{tr}(AB^\top).$$

This is true, since diagonal elements of $AB^\top$ are the following:

$$\begin{aligned} (1,1): \quad & a_{11}b_{11} + a_{12}b_{12} + \dots + a_{1n}b_{1n}; \\ (2,2): \quad & a_{21}b_{21} + a_{22}b_{22} + \dots + a_{2n}b_{2n}; \\ & \dots \\ (m,m): \quad & a_{m1}b_{m1} + a_{m2}b_{m2} + \dots + a_{mn}b_{mn}. \end{aligned}$$

$\boldsymbol{P_n(t)}$**,** $\boldsymbol{P(t)}$**,** $\boldsymbol{C[0,1]}$ Here we're considering the spaces of polynomials and the space of continuous functions on the interval $[0, 1]$. If f, g are 2 functions (or polynomials) we can define their scalar product by the following formula:

$$\langle f, g \rangle = \int_0^1 f(t)g(t)\,dt.$$

Moreover, we can consider the space of all functions which are continuous on the interval $[a, b]$, and in this case the scalar product will be defined as

$$\langle f, g \rangle = \int_a^b f(t)g(t)\,dt.$$

In all mentioned cases it is simple to check that the properties of the scalar product are satisfied.

2 Norm. Cauchy-Bunyakovsky-Schwartz inequality

In the vector space we can define the function, which is the natural generalization of the length of the vector.

Definition 2.1. *For any vector v from the Euclidean space we can define the number $\|v\|$, which is called the* ***norm*** *of v by the following formula:*

$$\|v\| = \sqrt{\langle v, v\rangle}$$

In the case of the space $\mathbb{R}^n$ we see that the norm of a vector is its length:

$$\|(x_1, x_2, \dots, x_n)\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

The following important fact about the scalar product is one of the most fundamental theorems in mathematics.

Theorem 2.2 (Cauchy-Bunyakovsky-Schwartz inequality). *For any vectors u and v from the Euclidean space V,*

$$\langle u, v\rangle^2 \leq \langle u, u\rangle\langle v, v\rangle.$$

In the different form this inequality can be written as

$$|\langle u, v\rangle| \leq \|u\|\|v\|.$$

Proof. Let t be an arbitrary number. Let's consider the scalar product of $tu + v$ with itself:

$$\begin{aligned}\langle tu + v, tu + v\rangle &= t^2\langle u, u\rangle + t\langle u, v\rangle + t\langle v, u\rangle + \langle v, v\rangle \\ &= t^2\langle u, u\rangle + 2t\langle u, v\rangle + \langle v, v\rangle\end{aligned}$$

This can be considered as a polynomial of the second power of t. Since $\langle tu + v, tu + v\rangle \geq 0$, then this quadratic polynomial is greater or equal to 0. So, it has either 1 on no roots, and thus it's discriminant is less then or equal to 0:

$$D = \langle u, v\rangle^2 - \langle u, u\rangle\langle v, v\rangle \leq 0.$$

Thus,

$$\langle u, v\rangle^2 \leq \langle u, u\rangle\langle v, v\rangle.$$

□

Using this theorem we can now consider the properties of the norm.

Positivity $\|v\| \geq 0$. Moreover, $\|v\| = 0$ if and only if $v = \mathbf{0}$.

Linearity $\|kv\| = |k|\|v\|$.

Triangle inequality $\|u + v\| \leq \|u\| + \|v\|$.

Proof. **Positivity:** Directly follows from the definition of the norm.

Linearity: $\|kv\| = \sqrt{\langle kv, kv\rangle} = \sqrt{k^2\langle v, v\rangle} = |k|\sqrt{\langle v, v\rangle} = |k|\|v\|$.

Triangle inequality: Let's consider $\|u + v\| \geq 0$. We can rewrite it in the following way:

$$\begin{aligned}
\|u+v\|^2 &= \langle u+v, u+v\rangle \\
&= \langle u, u\rangle + 2\langle u, v\rangle + \langle v, v\rangle \\
&= \|u\|^2 + 2\langle u, v\rangle + \|v\|^2 \qquad \text{by C-B-S inequality} \\
&\leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2 \\
&= (\|u\| + \|v\|)^2
\end{aligned}$$

Now, taking roots of both sides we get

$$\|u + v\| \leq \|u\| + \|v\|.$$

□

Geometrically speaking, the last inequality means that the side of the triangle is less then or equal to the sum of other 2 sides: