Lecture 22

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1 Properties of determinants

This lecture we will start studying a properties of determinants, and algorithms of computing them. Let's recall, that we defined a determinant by the following way:

$$\det \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} = \sum_{\substack{\text{all permutations of} \\ n \text{ elements } \sigma}} \operatorname{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}. \tag{1}$$

Now we'll start with properties of determinants.

Theorem 1.1 (1st elementary row operation). *If 2 rows of a matrix A are interchanged, then the determinant changes its sign.*

Proof. Suppose B arises from A by interchanging rows r and s of A, and suppose $r < s$. Then we have that $b_{rj} = a_{sj}$ and $b_{sj} = a_{rj}$ for any j, and $a_{ij} = b_{ij}$ if $i \neq r, s$. Now

$$\begin{aligned} \det B &= \sum_{\text{all permutations of } n \text{ elements } \sigma} \operatorname{sgn}(\sigma) b_{1\sigma(1)} \cdots b_{r\sigma(r)} \dots b_{s\sigma(s)} \dots b_{n\sigma(n)} \\ &= \sum_{\text{all permutations of } n \text{ elements } \sigma} \operatorname{sgn}(\sigma) a_{1\sigma(1)} \cdots a_{s\sigma(r)} \dots a_{r\sigma(s)} \dots a_{n\sigma(n)} \\ &= \sum_{\text{all permutations of } n \text{ elements } \sigma} \operatorname{sgn}(\sigma) a_{1\sigma(1)} \cdots a_{r\sigma(s)} \dots a_{s\sigma(r)} \dots a_{n\sigma(n)}. \end{aligned}$$

The permutation $(\sigma(1) \dots \sigma(s) \dots \sigma(r) \dots \sigma(n))$ is obtained from $(\sigma(1) \dots \sigma(r) \dots \sigma(s) \dots \sigma(n))$ by interchanging 2 numbers, so its sign is different, and $\det B = -\det A$. □

Theorem 1.2 (Determinant of a matrix with 2 equal rows). *If 2 rows of a matrix are equal, then its determinant is equal to 0.*

Proof. Suppose rows r and s of matrix A are equal. Interchange them to obtain matrix B. Then $\det B = -\det A$. On the other hand, $B = A$, so $\det B = \det A$. So, $\det A = -\det A$, and thus $\det A = 0$. □

Theorem 1.3 (2nd elementary row operation). *If B is obtained from A by multiplying a row of A by a real number c, then $\det B = c \det A$.*

Proof. Suppose r-th row of A is multiplied by c to obtain B. Then $b_{rj} = ca_{rj}$ for any j and $b_{ij} = a_{ij}$ if $i \neq r$. Thus

$$\begin{aligned}
\det B &= \sum_{\text{all permutations of } n \text{ elements } \sigma} \operatorname{sgn}(\sigma) b_{1\sigma(1)} \cdots b_{r\sigma(r)} \dots b_{n\sigma(n)} \\
&= \sum_{\text{all permutations of } n \text{ elements } \sigma} \operatorname{sgn}(\sigma) a_{1\sigma(1)} \cdots (ca_{r\sigma(r)}) \dots a_{n\sigma(n)} \\
&= c \cdot \sum_{\text{all permutations of } n \text{ elements } \sigma} \operatorname{sgn}(\sigma) a_{1\sigma(1)} \cdots a_{r\sigma(r)} \dots a_{n\sigma(n)} \\
&= c \det A.
\end{aligned}$$

□

Theorem 1.4. *If a row of a matrix A consists entirely of zeros, then $\det A = 0$.*

Proof. Let's multiply the zero row of a matrix A by a nonzero number c to obtain matrix B. Then $\det B = c \det A$, But $B = A$, so $\det B = \det A$, and thus $\det A = c \det A$. So, $\det A = 0$. □

Theorem 1.5 (Multilinearity by rows). *If in matrix A row a_r can be represented as sum of rows b and c, i.e. $a_{rj} = b_j + c_j$, i.e.*

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \multicolumn{4}{c}{\dots\dots\dots\dots\dots\dots} \\ a_{r1} & a_{r2} & \dots & a_{rn} \\ \multicolumn{4}{c}{\dots\dots\dots\dots\dots\dots} \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \multicolumn{4}{c}{\dots\dots\dots\dots\dots\dots} \\ b_1 + c_1 & b_2 + c_2 & \dots & b_n + c_n \\ \multicolumn{4}{c}{\dots\dots\dots\dots\dots\dots} \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

then

$$\det \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \multicolumn{4}{c}{\dots\dots\dots\dots\dots\dots} \\ b_1 + c_1 & b_2 + c_2 & \dots & b_n + c_n \\ \multicolumn{4}{c}{\dots\dots\dots\dots\dots\dots} \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} = \det \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \multicolumn{4}{c}{\dots\dots\dots\dots\dots\dots} \\ b_1 & b_2 & \dots & b_n \\ \multicolumn{4}{c}{\dots\dots\dots\dots\dots\dots} \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} + \det \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \multicolumn{4}{c}{\dots\dots\dots\dots\dots\dots} \\ c_1 & c_2 & \dots & c_n \\ \multicolumn{4}{c}{\dots\dots\dots\dots\dots\dots} \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

Formally, this property tells us that the determinant is a multilinear function of rows of a matrix.

Proof. We'll use the definition of the determinant.

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \multicolumn{4}{c}{\dotfill} \\ b_1 + c_1 & b_2 + c_2 & \dots & b_n + c_n \\ \multicolumn{4}{c}{\dotfill} \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} = \sum_{\text{all perms of } n \text{ elems } \sigma} \operatorname{sgn}(\sigma) a_{1\sigma(1)} \cdots (b_{\sigma(r)} + c_{\sigma(r)}) \dots a_{n\sigma(n)}$$

$$= \sum_{\text{all perms of } n \text{ elems } \sigma} \operatorname{sgn}(\sigma) a_{1\sigma(1)} \cdots b_{\sigma(r)} \dots a_{n\sigma(n)}$$

$$+ \sum_{\text{all perms of } n \text{ elems } \sigma} \operatorname{sgn}(\sigma) a_{1\sigma(1)} \cdots c_{\sigma(r)} \dots a_{n\sigma(n)}$$

$$= \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \multicolumn{4}{c}{\dotfill} \\ b_1 & b_2 & \dots & b_n \\ \multicolumn{4}{c}{\dotfill} \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \multicolumn{4}{c}{\dotfill} \\ c_1 & c_2 & \dots & c_n \\ \multicolumn{4}{c}{\dotfill} \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

□

Theorem 1.6 (3rd elementary row operation). *If B is obtained from A by adding a row r multiplied by c to row s, then* $\det B = \det A$.

Proof.

$$\begin{vmatrix} \multicolumn{3}{c}{\dotfill} \\ a_{r1} & \dots & a_{rn} \\ \multicolumn{3}{c}{\dotfill} \\ a_{s1} + ca_{r1} & \dots & a_{sn} + ca_{rn} \\ \multicolumn{3}{c}{\dotfill} \end{vmatrix} = \begin{vmatrix} \multicolumn{3}{c}{\dotfill} \\ a_{r1} & \dots & a_{rn} \\ \multicolumn{3}{c}{\dotfill} \\ a_{s1} & \dots & a_{sn} \\ \multicolumn{3}{c}{\dotfill} \end{vmatrix} + \begin{vmatrix} \multicolumn{3}{c}{\dotfill} \\ a_{r1} & \dots & a_{rn} \\ \multicolumn{3}{c}{\dotfill} \\ ca_{r1} & \dots & ca_{rn} \\ \multicolumn{3}{c}{\dotfill} \end{vmatrix}$$

$$= \det A + c \begin{vmatrix} \multicolumn{3}{c}{\dotfill} \\ a_{r1} & \dots & a_{rn} \\ \multicolumn{3}{c}{\dotfill} \\ a_{r1} & \dots & a_{rn} \\ \multicolumn{3}{c}{\dotfill} \end{vmatrix}$$

$$= \det A$$

since the determinant of the last matrix is equal to 0, because its rows are equal. □

Theorem 1.7 (Determinant of a triangular matrix). *The determinant of a triangular matrix is equal to the product of its diagonal elements.*

Proof. The product of diagonal elements is included into he expression for the determinant, and its sign is "+". All other terms are equal to 0, if the matrix is triangular. Let's prove it.

Let $a_{1k_1} a_{2k_2} \dots a_{nk_n} \neq 0$. Then

$$k_1 \geq 1,\ k_2 \geq 2\ \dots,\ k_n \geq n$$

(otherwise the term is equal to 0). But $(k_1 k_2 \dots k_n)$ is a permutation of numbers from 1 to n, so

$$k_1 + k_2 + \cdots + k_n = 1 + 2 + \cdots + n,$$

and it is possible only if

$$k_1 = 1,\ k_2 = 2,\ \dots,\ k_n = n.$$

$\square$

So, now we know what happens with the determinant after applying elementary row operations. So, we can now give the algorithm of computing the determinant.

Algorithm. Transform matrix A by elementary row operation to the triangular form keeping track of how the determinant changes.

Example 1.8.

$$\det \begin{pmatrix} 4 & 3 & 2 \\ 3 & -2 & 5 \\ 2 & 4 & 6 \end{pmatrix} = 2 \det \begin{pmatrix} 4 & 3 & 2 \\ 3 & -2 & 5 \\ 1 & 2 & 3 \end{pmatrix} \qquad \textit{div. 3rd row by 2}$$

$$= -2 \det \begin{pmatrix} 1 & 2 & 3 \\ 3 & -2 & 5 \\ 4 & 3 & 2 \end{pmatrix} \qquad \textit{interchange rows 1 and 3}$$

$$= -2 \det \begin{pmatrix} 1 & 2 & 3 \\ 0 & -8 & -4 \\ 4 & 3 & 2 \end{pmatrix} \qquad \textit{mult. 1st row by 3 and sub. from 2nd}$$

$$= -2 \det \begin{pmatrix} 1 & 2 & 3 \\ 0 & -8 & -4 \\ 0 & -5 & -10 \end{pmatrix} \qquad \textit{mult. 1st row by 4 and sub. from 3rd}$$

$$= (-2)(4) \det \begin{pmatrix} 1 & 2 & 3 \\ 0 & -2 & -1 \\ 0 & -5 & -10 \end{pmatrix} \qquad \textit{div. 2nd row by 4}$$

$$= (-2)(4)(5) \det \begin{pmatrix} 1 & 2 & 3 \\ 0 & -2 & -1 \\ 0 & -1 & -2 \end{pmatrix} \qquad \textit{div. 3rd row by 5}$$

$$= (-2)(4)(5) \det \begin{pmatrix} 1 & 2 & 3 \\ 0 & -2 & -1 \\ 0 & 0 & -\frac{3}{2} \end{pmatrix} \qquad \textit{mult. the 2nd row by 1/2 and sub. from 3rd}$$

$$= (-2)(4)(5)(1)(-2)(-\frac{3}{2}) = -120.$$