Lecture 20

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1 Permutations

Let we have n numbers from 1 to n: $1, 2, \ldots, n$. If we change their order we get their **permutation**. We will write these numbers in brackets, for example, (34152) is a permutation of numbers from 1 to 5.

Example 1.1. *There are 6 different permutations of numbers* $1, 2$ *and* 3*:*

$$\begin{array}{cc} (123) & (132) \\ (213) & (231) \\ (312) & (321) \end{array}$$

Let's compute the total number of permutations of numbers from 1 to n. On the first place in the permutation we can put any of n elements. On the second place we can put any element which is not equal to the element on the first place — so we have $(n-1)$ possibilities; than to the third place we can put any element which is not on the first and the second place, so totally we have $(n-2)$ possibilities, etc. Finally, we will have only one possibility to put an element on the last n-th place. So, total number of permutations of n elements is $n(n-1)(n-2)\cdots 2\cdot 1 = n!$ (By definition, $n!$ is a product of all numbers from 1 to n. It is called n-**factorial**).

So, we see that the number of permutations increases very fast. We have just $3! = 6$ different permutations of numbers from 1 to 3, $4! = 24$ different permutations of numbers from 1 to 4, $5! = 120$ different permutations of numbers from 1 to 5, and the number of permutations of the numbers from 1 to 10 is already equal to $10! = 3628800$.

We will use the following notation. The permutations will be denoted by Greek letters σ and τ — "sigma" and "tau" respectively. For example, we will write $\sigma = (3241)$ and $\tau = (231)$. The set of all permutations from 1 to n will be denoted by S_n. For example, as we saw already, $S_3 = \{(123), (132), (213), (231), (312), (321)\}$.

By $\sigma(i)$ we will denote the i-th element of the permutation σ. For example, if $\sigma = (3241)$, then $\sigma(1) = 3$, $\sigma(2) = 2$, $\sigma(3) = 4$, and $\sigma(4) = 1$.

Definition 1.2. *Two elements of the permutation form an **inversion** if the largest stands to the left of the smallest.*

Definition 1.3. *The permutation is called* ***even*** *if the total number of inversions is even, and* ***odd*** *if the total number of inversions is odd.*

The number $(-1)^{\#\text{ of inversions}}$ *is called the* ***sign of the permutation****, and is equal to 1 for even permutations and -1 for odd. It is denoted by* $\operatorname{sgn}(\sigma)$.

Example 1.4.

- $\sigma = (123)$. *There are no inversions at all. So, it is even, and* $\operatorname{sgn}(\sigma) = 1$.
- $\sigma = (132)$. *Numbers 3 and 2 form an inversion. So, there is only 1 inversion. So, it is add, and* $\operatorname{sgn}(\sigma) = -1$.
- $\sigma = (213)$. *Numbers 2 and 1 form an inversion. So, there is only one inversion. So, it is odd, and* $\operatorname{sgn}(\sigma) = -1$.
- $\sigma = (231)$. *Numbers 2 and 1 form an inversion. Numbers 3 and 1 form an inversion. So, totally there are 2 inversions. So, it is even, and* $\operatorname{sgn}(\sigma) = 1$.
- $\sigma = (312)$. *Numbers 3 and 1 form an inversion. Numbers 3 and 2 form an inversion. So, totally there are 2 inversions. So, it is even, and* $\operatorname{sgn}(\sigma) = 1$.
- $\sigma = (321)$. *Numbers 3 and 2 form an inversion. Numbers 3 and 1 form an inversion. Numbers 2 and 1 form an inversion. So, there are 3 inversions. So, it is odd, and* $\operatorname{sgn}(\sigma) = -1$.

The main fact about permutations is the following.

Lemma 1.5. *If we interchange any 2 elements in the permutation, its sign changes, i.e. if it was even, it becomes odd, and other way round.*

Example 1.6. *Let* $\sigma = (25431)$. *The following pairs of numbers form an inversion: 21, 54, 53, 51, 43, 41, 31. So, we have 7 inversions, and this permutation is odd. Let's interchange 5 and 1. We'll get* $\tau = (21435)$. *The following pairs of numbers form an inversion: 21, 43. So, we have just 2 inversions, so this permutation is even.*

Proof of the lemma. First let's note, that if we transpose 2 consecutive elements, the number of inversions changes exactly by 1. So, the parity of the permutation changes. Transposition of the elements i and j can be done by $2s+1$ consecutive transpositions of the adjacent elements: first interchange i with all elements on its way to j, and than move j to the place where we had i. So, the sign will change odd number of times, and the parity of the permutation will change. □

We'll demonstrate this proof by an example. Let we want to change 1 and 5 in the permutation (25431). The arrow will denote a pair of consecutive elements which should be interchanged.

1. $(25 \leftrightarrow 431)$. Inversions: 21, 54, 53, 51, 43, 41, 31.
2. $(245 \leftrightarrow 31)$. Inversions: 21, 43, 41, 53, 51, 31.
3. $(2435 \leftrightarrow 1)$. Inversions: 21, 43, 41, 31, 51.
4. $(243 \leftrightarrow 15)$. Inversions: 21, 43, 41, 31.
5. $(24 \leftrightarrow 135)$. Inversions: 21, 41, 43.
6. (21435). Inversions: 21, 43.

So, there were 5 transpositions, so the sign changed 5 times. The starting permutation is odd, and the final is even.

There is another important fact about permutations.

Lemma 1.7. *For any number n the number of odd permutations of numbers from 1 to n is equal to the number of even permutations.*

Proof. If we have an even permutation, then after transposition of the first 2 elements we'll get an odd permutation. So, we'll get all permutations. □