Lecture 19

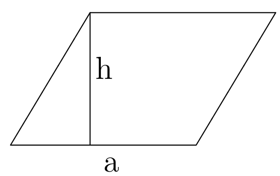
Andrei Antonenko

March 24, 2003

1 Area of the parallelogram

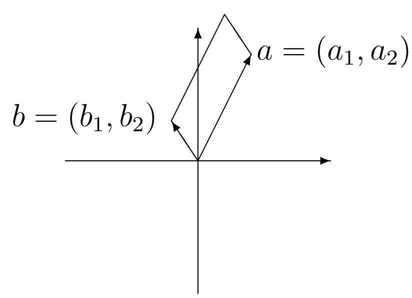
Let's consider a plane $\mathbb{R}^2$. Now we will consider parallelograms on this plane, and compute their area.

First thing which is clear from elementary geometry is a formula for the area of the parallelogram.



The area of the parallelogram is equal to the product of the base band the height, $S = ah$.

Now let's consider a plane $\mathbb{R}^2$ as a vector space, and let we have 2 vectors $a = (a_1, a_2)$ and $b = (b_1, b_2)$ on the plane.



Now with this pair of vectors we can associate a parallelogram, as shown on the picture above. Out main goal is to study the properties of the area of this parallelogram and compute it in terms of vectors $a = (a_1, a_2)$ and $b = (b_1, b_2)$.

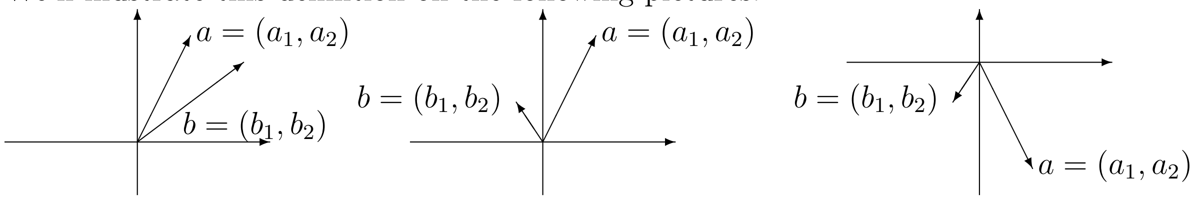
First let's give a definition of the oriented area of the parallelogram.

Definition 1.1. *The* ***oriented area***

$$\text{area}(a, b)$$

of the parallelogram based on two vectors $a = (a_1, a_2)$ and $b = (b_1, b_2)$ is the standard geometrical area of it taken with appropriate sign. The sign is determined by the following rule. If the rotation from a to b (by the smaller angle) goes counterclockwise, then the sign is "+", otherwise, the sign is "−".

We'll illustrate this definition on the following pictures.



From a to b we're rotating in clockwise direction, so sign is "−".

From a to b we're rotating in counterclockwise direction, so sign is "+".

From a to b we're rotating in clockwise direction, so sign is "−".

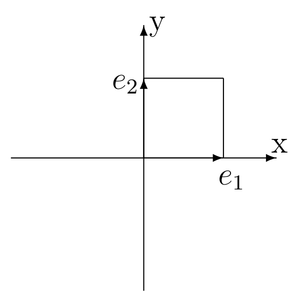
Now we'll state properties of the oriented area.

1. $\operatorname{area}(a, b) = -\operatorname{area}(b, a)$. This property follows from the fact that if we turn from a to b clockwise, then from b to a we turn counterclockwise and other way round, so signs are different, and absolute values will be the same since the parallelogram doesn't change.

   From this property it follows that $\operatorname{area}(a, a) = 0$ for any a:

$$\operatorname{area}(a, a) = -\operatorname{area}(a, a) \quad \Leftrightarrow \quad \operatorname{area}(a, a) = 0.$$

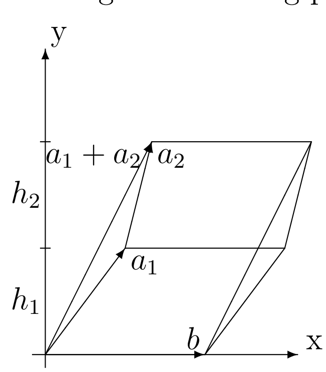
2. $\operatorname{area}(e_1, e_2) = 1$. This is an area of the unit square, so it is equal to 1:



3. (a)

$$\operatorname{area}(a_1 + a_2, b) = \operatorname{area}(a_1, b) + \operatorname{area}(a_2, b).$$

   This property we can illustrate using the following picture.



   So, $\operatorname{area}(a_1, b) = -h_1 b$, then $\operatorname{area}(a_2, b) = -h_2 b$, and $\operatorname{area}(a_1 + a_2, b) = -(h_1 + h_2)b$, so this property holds.

(b) In the same way it's possible to see that the property

$$\text{area}(a, b_1 + b_2) = \text{area}(a, b_1) + \text{area}(a, b_2).$$

4. For any number k

$$\text{area}(ka, b) = k\,\text{area}(a, b);$$
$$\text{area}(a, kb) = k\,\text{area}(a, b).$$

Now we can use the following properties to calculate the area of a parallelogram. Let $a = (a_1, a_2)$ and $b = (b_1, b_2)$. Then $a = a_1(1, 0) + a_2(0, 1) = a_1e_1 + a_2e_2$ and $b = b_1(1, 0) + b_2(0, 1) = b_1e_1 + b_2e_2$. Now

$$\begin{aligned}
\text{area}(a, b) &= \text{area}(a_1e_1 + a_2e_2, b_1e_1 + b_2e_2) \\
&= \text{area}(a_1e_1, b_1e_1 + b_2e_2) + \text{area}(a_2e_2, b_1e_1 + b_2e_2) \\
&= \text{area}(a_1e_1, b_1e_1) + \text{area}(a_1e_1, b_2e_2) + \text{area}(a_2e_2, b_1e_1) + \text{area}(a_2e_2, b_2e_2) \\
&= a_1b_1\,\text{area}(e_1, e_1) + a_1b_2\,\text{area}(e_1, e_2) + a_2b_1\,\text{area}(e_2, e_1) + a_2b_2\,\text{area}(e_2, e_2) \\
&= a_1b_2 - a_2b_1
\end{aligned}$$

Now let's write the coordinates of vectors as rows of a matrix. We'll have the following definition.

Definition 1.2. ***Determinant*** *of a* 2×2*-matrix* $\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$ *is defined as following:*

$$\det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} = a_1b_2 - a_2b_1.$$

So, geometrically the determinant represents an area of the parallelogram based on vectors (a_1, a_2) and (b_1, b_2).

Example 1.3.

$$\det \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} = \cos\alpha \cdot \cos\alpha - (-\sin\alpha)\sin\alpha = \cos^2\alpha + \sin^2\alpha = 1.$$

In the same way we can try to determine the volume of the parallelepiped in the 3-dimensional space. We can derive the similar formula for the volume, if its edges are vectors (a_1, a_2, a_3), (b_1, b_2, b_3), and (c_1, c_2, c_3). If we write these vectors as rows of a matrix, the volume is called the determinant of this matrix.

Definition 1.4. ***Determinant*** *of a* 3×3*-matrix* $\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$ *is defined as following:*

$$\det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3.$$

Geometrically, determinant of a 3×3-matrix is the volume of the parallelepiped based on vectors (a_1, a_2, a_3), (b_1, b_2, b_3), and (c_1, c_2, c_3).

We'll give a mnemonic rule of writing determinants of a 3×3-matrix. We will rewrite first two columns of a matrix at the end of it, and then take following products with "+" signs:

$$\begin{array}{ccccc} a_1 & a_2 & a_3 & a_1 & a_2 \\ b_1 & b_2 & b_3 & b_1 & b_2 \\ c_1 & c_2 & c_3 & c_1 & c_2 \end{array}$$

and following products with "−" signs:

$$\begin{array}{ccccc} a_1 & a_2 & a_3 & a_1 & a_2 \\ b_1 & b_2 & b_3 & b_1 & b_2 \\ c_1 & c_2 & c_3 & c_1 & c_2 \end{array}$$

Example 1.5.

$$\begin{aligned} \det \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} &= 1 \cdot 5 \cdot 9 + 2 \cdot 6 \cdot 7 + 3 \cdot 4 \cdot 8 - 3 \cdot 5 \cdot 7 - 1 \cdot 6 \cdot 8 - 2 \cdot 4 \cdot 9 \\ &= 45 + 84 + 96 - 105 - 48 - 72 \\ &= 225 - 225 \\ &= 0 \end{aligned}$$

Geometrically this result means that vectors $(1, 2, 3)$, $(4, 5, 6)$, *and* $(7, 8, 9)$ *are in the same plane, so the volume of a parallelepiped based on them is equal to 0.*