Lecture 9

Andrei Antonenko

February 21, 2003

1 Linear combinations

Definition 1.1. *Let V be a vector space. A vector $v \in V$ is a* ***linear combination*** *of vectors $u_1, u_2, \dots, u_n$ if there exist $a_1, a_2, \dots, a_n \in \Bbbk$ such that*

$$v = a_1u_1 + a_2u_2 + \cdots + a_nu_n. \tag{1}$$

Sometimes it is possible to express a vector as a linear combination of other vectors. It can be done by solving a corresponding linear system. We'll demonstrate it in the following example.

Example 1.2. *Consider the space $\mathbb{R}^2$ — the space of all pairs of numbers. Let $v = (8, 13)$, $u_1 = (1, 2)$, and $u_2 = (2, 3)$. Let's express v as a linear combination of u_1 and u_2. To do this we have to find a and b such that $v = au_1+bu_2$, i.e. $(8, 13) = a(1, 2)+b(2, 3) = (a\cdot1+b\cdot2, a\cdot2+b\cdot3)$. So, we get the following system:*

$$\begin{cases} 1a + 2b = 8 \\ 2a + 3b = 13 \end{cases}$$

We can simply solve this system: subtracting the first equation multiplied by 2 from the second one we get $-b = -3$, so $b = 3$, and so $a = 8-2b = 2$. So we see that $(8, 13) = 2\cdot(1, 2)+3\cdot(2, 3)$.

Example 1.3. *Consider the space $P(t)$ — space of all polynomials. Let $v = 5t^2 + 2t + 1$, $u_1 = t^2 + t$, $u_2 = t + 1$, $u_3 = t^2 + 1$. Let's express v as a linear combination of u_1, u_2 and u_3. We should find a, b and c such that $v = au_1 + bu_2 + cu_3$, i.e. $5t^2 + 2t + 1 = a(t^2 + t) + b(t + 1) + c(t^2 + 1) = t^2(a + c) + t(a + b) + (b + c)$. So, we get the following system:*

$$\begin{cases} a + c = 5 \\ a + b = 2 \\ b + c = 1 \end{cases}$$

Let's solve this system. Subtracting the first equation from the second one we get

$$\begin{cases} a \quad\quad + c = 5 \\ \quad b - c = -3 \\ \quad b + c = 1 \end{cases}$$

and than subtracting the second from the third one we get

$$\begin{cases} a \quad\quad + c = 5 \\ \quad b - c = -3 \\ \quad\quad\quad 2c = 4 \end{cases}$$

So, $c = 2$, $b = -3 + 2 = -1$, *and* $a = 5 - 2 = 3$. *So,* $5t^2 + 2t + 1 = 3(t^2 + t) - (t + 1) + 2(t^2 + 1)$.

2 Linear dependence and independence

Now we'll study one of the most important concepts of linear algebra and the theory of vector spaces. This is a concept of linear dependence and independence.

Definition 2.1. *Let* $u_1, u_2, \ldots, u_n$ *be a system of vectors. A linear combination of them is called* ***nontrivial*** *if there exists a nonzero coefficient. If all coefficients are equal to 0, the linear combination is called* ***trivial****.*

Example 2.2. $u_1 + 0u_2 + 0u_3 - 3u_4$ *is nontrivial linear combination, and* $0u_1 + 0u_2 + 0u_3 + 0u_4$ *is a trivial linear combination.*

Definition 2.3. *A system of vectors* $u_1, u_2, \ldots, u_n$ *is called* ***linearly dependent*** *if there exists a nontrivial linear combination of these vectors which is equal to zero. Otherwise the system is called* ***linearly independent****.*

Example 2.4. *Consider a vector space* $\mathbb{R}^3$. *Let* $u_1 = (3, -5, 0)$, $u_2 = (5, 0, 1)$, *and* $u_3 = (8, -5, 1)$. *Then linear combination with coefficients 1,1, and -1 is nontrivial and equals to zero:*

$$1 \cdot (3, 5, 0) + 1 \cdot (5, 0, 1) + (-1) \cdot (8, -5, 1) = (0, 0, 0).$$

Example 2.5. *Consider a vector space* $\mathbb{R}^2$. *Let* $u_1 = (1, 1)$, *and* $u_2 = (0, 0)$. *The linear combination with coefficients 0 and 1 is nontrivial, and equals to zero:*

$$0 \cdot (1, 1) + 1 \cdot (0, 0) = (0, 0)$$

Moreover, if one of the vectors in the system equals to $\mathbf{0}$, *then this system is linearly dependent, since we can make a coefficient before it equal to some nonzero number, and all other coefficients we can make equal to zero.*

Example 2.6. *Consider the space $\mathbb{R}^3$. Let's figure out if the vectors $u_1 = (1,0,0)$, $u_2 = (0,1,0)$ and $u_3 = (0,0,1)$ are linearly dependent. We need to have $au_1 + bu_2 + cu_3 = 0$, i.e.*

$$a\begin{pmatrix}1\\0\\0\end{pmatrix} + b\begin{pmatrix}0\\1\\0\end{pmatrix} + c\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

This leads to the following system of equations:

$$\begin{cases} a & & & = & 0 \\ & b & & = & 0 \\ & & c & = & 0 \end{cases}$$

This system has only one solution — $(0,0,0)$. So, if we have a linear combination which is equal to 0, then it is trivial. So, this system of vectors is independent.

Example 2.7. *Consider the space $\mathbb{R}^3$. Let's figure out if the vectors $u_1 = (1,1,1)$, $u_2 = (1,1,0)$ and $u_3 = (1,0,0)$ are linearly dependent. We need to have $au_1 + bu_2 + cu_3 = 0$, i.e.*

$$a\begin{pmatrix}1\\1\\1\end{pmatrix} + b\begin{pmatrix}1\\1\\0\end{pmatrix} + c\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

This leads to the following system of equations:

$$\begin{cases} a + b + c = 0 \\ a + b = 0 \\ a = 0 \end{cases}$$

This system has only one solution — $(0,0,0)$. So, if we have a linear combination which is equal to 0, then it is trivial. So, this system of vectors is independent.

Now we'll state a lemma about linear dependence.

Lemma 2.8. *Vectors $u_1,\ u_2,\dots,u_n$ are linearly dependent if and only if some of them can be expressed as a linear combination of others.*

Proof. Let $u_1,\ u_2,\dots,u_n$ are linearly dependent, i.e. there exists nontrivial linear combination

$$a_1u_1 + a_2u_2 + \cdots + a_nu_n = 0.$$

It is nontrivial, so at least one of coefficients, say, a_1, is not equal to 0. Then

$$u_1 = -\frac{a_2}{a_1}u_2 - \frac{a_3}{a_1}u_3 - \cdots - \frac{a_n}{a_1}u_n$$

so, u_1 is expressed as a linear combination of other vectors.

On the contrary, let one of these vectors, say, u_1, can be expressed as a linear combination of other vectors:

$$u_1 = b_2u_2 + b_3u_3 + \cdots + b_nu_n.$$

Then

$$u_1 - b_2u_2 - b_3u_3 - \cdots - b_nu_n = 0$$

is a nontrivial linear combination which is equal to zero, and so vectors are linearly dependent.

□

Example 2.9. *The vectors* $u_1 = (0, 3, 5)$, $u_2 = (-1, 4, 7)$ *and* $u_3 = (1, 2, 3)$ *are linearly dependent since the linear combination with coefficients -2, 1 and 1 is equal to 0:*

$$-2(0, 3, 5) + (-1, 4, 7) + (1, 2, 3) = (0, 0, 0).$$

So, vector u_1 *can be expressed as a linear combination of* u_2 *and* u_3*:*

$$u_1 = (0, 3, 5) = \frac{1}{2}(-1, 4, 7) + \frac{1}{2}(1, 2, 3)$$

Example 2.10. *Let*

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Here u_1, u_2 *and* u_3 *are linearly independent and none of these vectors can be expressed as a linear combination of other 2 vectors. For example, for* u_1 *there are no real* a *and* b *such that*

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = au_2 + bu_3 = a\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$